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ON SOME PROPERTIES OF K-MONOTONE OPERATORS

RADU PRECUP
(Cluj-Napoca)

1. Introduction. In an earlier paper [14], we have introduced the class of K-monotone operators, which is wider than the class of monotone (in the sense of Minty-Browder) operators. Since this class of operators also includes a sufficiently large set of (o)-monotone (monotone in the sense of order) operators, it follows that the investigation of K-monotone operators is of interest at least for a unitary approach to the theories of monotone and (o)-monotone operators. The fact that this unitary approach is a natural one, follows from refs. [15], [16], where the best-approximation operator of the elements of a Hilbert space X, by elements of a nonvoid closed convex subset C and with respect to a certain norm on X, is investigated. In some conditions imposed to the norm and to the subset C, the best-approximation operator can be monotone, or (o)-monotone, or (essentially) only K-monotone. In this paper, we shall extend some well-known basic results in monotone operators to the class of K-monotone operators. Part of these extensions were already done in [14].

2. K-Monotone operators. Let X and Y be two real linear spaces. By $\langle .,. \rangle$ we denote a bilinear functional on $X \times Y$. If $K \subset X$ is a *convex cone* (i.e. $K + K \subset K$ and $\alpha K \subset K$ for any $\alpha \geqslant 0$), then the *polar cone* K^* of K with respect to the bilinear functional $\langle .,. \rangle$ is defined by $K^* = \{y \in Y;\ \langle x, y \rangle \geqslant 0$ for all $x \in K\}$. Let $A : X \to 2^Y$ be a multivalued operator and denote by $D(A)$ the set $\{x \in X;\ Ax \neq \emptyset\}$. The operator A is called *monotone* (with respect to the bi-linear functional $\langle .,. \rangle$) if for any $x,\ x' \in D(A)$, the inequality $\langle x - x',\ y - y' \rangle \geqslant 0$ holds for all $y \in Ax$ and $y' \in Ax'$. The operator A is said to be *(o)-monotone* (monotone in the sense of order) provided that whenever $x,\ x' \in D(A)$ and $x - x' \in K$, then $y - y' \in K^*$ for all $y \in Ax$ and $y' \in Ax'$. In ref. [14], the operator A was called *K-monotone* provided that for any $x,\ x' \in D(A)$ such that $x - x' \in K$, the inequality $\langle x - x',\ y - y' \rangle \geqslant 0$ holds for all $y \in Ax$ and $y' \in Ax'$. A monotone ((o)-monotone or K-monotone) operator A is said to be *maximal monotone* (*maximal (o)-monotone* or *maximal K-monotone*) if, whenever B is an operator having the same property as A and $Ax \subset Bx$ for all $x \in X$, then $A = B$.

It is clear that an operator is K-monotone if and only if it is $(-K)$-monotone. Also, each monotone operator is K-monotone for any convex cone K and if $K \cup (-K) = X$, then the K-monotonicity reduces to monotonicity.

It is also evident that each (o)-monotone operator is K-monotone. Moreover, if $C \subset Y$ is a convex cone and $T : Y \to Y$ is a linear operator which maps C into K^*, i.e. $T(C) \subset K^*$, then each operator $A : X \to 2^Y$ which is (K, C)-monotone, in the sense that whenever for $x, x' \in D(A)$ one has $x - x' \in K$, then $y - y' \in C$ for all $y \in Ax$ and $y' \in Ax'$, is K-monotone with respect to the bilinear functional $\langle .,T(\cdot)\rangle$, i.e. the inequality $\langle x - x', T(y - y')\rangle \geqslant 0$ holds for every $x, x' \in D(A)$ satisfying $x - x' \in K$ and for all $y \in Ax$ and $y' \in Ax'$.

Also, if $L : X \to X$ is a linear operator which maps K into C^*, i.e. $L(K) \subset C^*$, where $C^* = \{x \in X;\ \langle x, y\rangle \geqslant 0$ for all $y \in C\}$, then each operator $A : X \to 2^Y$ which is (K, C)-monotone, is K-monotone with respect to the bi-linear functional $\langle L(.), .\rangle$, i.e. the inequality $\langle L(x - x'), y - y'\rangle \geqslant 0$ holds for every $x, x' \in D(A)$ satisfying $x - x' \in K$ and for all $y \in Ax$ and $y' \in Ax'$.

If, in addition, X and Y are separated locally convex spaces, $\bar{K} \neq X$ and $\bar{C} \neq Y$, then there exist [13, 2, 2.12] two non-trivial continuous linear functionals $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ such that $f(K) \subset [0, +\infty[$ and $g(C) \subset [0, +\infty[$. We can immediately see that each (K, C)-monotone operator $A : X \to 2^Y$ is K-monotone with respect to the bilinear functional defined by $\langle x, y\rangle = f(x)\, g(y)$ for $x \in X$ and $y \in Y$.

3. A maximality result on K-monotone operators. Throughout this paper, X will be a real linear normed space, X^* its dual X^* and the bi-linear functional on $X \times X^*$ will be the natural duality between X and X^*, that is $\langle x, x^*\rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$.

We shall say that the operator $A : D(A) \to X^*$, where $D(A) \subset X$, is *K-hemicontinuous at* $x \in D(A)$ if $\lim\limits_{t \downarrow 0} A(x + ty) = Ax$ with respect to the weak* topology of X^*, for any $y \in K \cup (-K)$ such that $x + ty \in D(A)$ for $0 \leqslant t \leqslant 1$. A is said to be *K-hemicontinuous* if it is K-hemicontinuous at each $x \in D(A)$. In the particular case when $K = X$, we shall employ the usual "hemicontinuity" term instead of "X-hemicontinuity".

LEMMA 3.1. *Let $K \subset X$ be a convex cone with non-empty interior and let $A : D(A) \to X^*$ be a K-hemicontinuous operator, where $D(A)$ is a dense convex subset of X. If $x_0 \in D(A)$, $x_0^* \in X^*$ and the inequality*

$$\langle x - x_0,\ Ax - x_0^*\rangle \geqslant 0 \tag{3.1}$$

holds for every $x \in D(A)$ such that $x - x_0 \in K \cup (-K)$, then $x_0^* = Ax_0$.

Proof. First, we mention that if $\operatorname{int} K \neq \emptyset$, then $\operatorname{int} K + (-\operatorname{int} K) = X$ and $\operatorname{int} K \cup \{0\}$ is a convex cone.

Suppose that $x_0^* \neq Ax_0$. Then, since $\operatorname{int} K + (-\operatorname{int} K) = X$, we may find x in X such that

$$x - x_0 \in \operatorname{int} K \cup (-\operatorname{int} K) \tag{3.2}$$

and

$$\langle x - x_0,\ Ax_0 - x_0^*\rangle < 0. \tag{3.3}$$

Since $D(A)$ is a dense subset of X and relations (3.2) and (3.3) are satisfied in a whole neighbourhood of x, we may assume that in relations (3.2) and (3.3) $x \in D(A)$. Next, the convexity of $D(A)$ implies that $x_t = x_0 + t(x - x_0) \in D(A)$ for $0 \leqslant t \leqslant 1$. Also, by (3.2) we have $x_t - x_0 = t(x - x_0) \in \operatorname{int} K \cup (-\operatorname{int} K) \subset K \cup (-K)$ for $0 < t \leqslant 1$. Consequently, x_t satisfies inequality (3.1), i.e. $\langle x_t - x_0,\ Ax_t - x_0^*\rangle \geqslant 0$, for $0 < t \leqslant 1$. It follows that $\langle x - x_0,\ Ax_t - x_0^*\rangle \geqslant 0$ for $0 < t \leqslant 1$. Passing to limit as $t \downarrow 0$ and taking into account the K-hemicontinuity of A, we obtain $\langle x - x_0,\ Ax_0 - x_0^*\rangle \geqslant 0$, which contradicts relation (3.3). Hence, $x_0^* = Ax_0$.

THEOREM 3.1. *If the convex cone $K \subset X$ has non-empty interior, then each K-hemicontinuous K-monotone operator $A : X \to X^*$ is maximal K-monotone.*

Proof. Let $B : X \to 2^{X^*}$ be a K-monotone operator such that $Ax \in Bx$ for all $x \in X$. Consider (arbitrarily) $x_0 \in X = D(A)$ and $x_0^* \in Bx_0$. As B is K-monotone, we have $\langle x - x_0,\ Ax - x_0^*\rangle \geqslant 0$ for all $x \in X$ such that $x - x_0 \in K \cup (-K)$. Hence, by Lemma 3.1, $x_0^* = Ax_0$. Thus, $Ax_0 = Bx_0$ for every $x_0 \in X$, which proves the maximality of A.

COROLLARY 3.1. *Let $A : X \to X^*$ be a continuous linear operator satisfying the inequality $\langle x, Ax\rangle > 0$ for at least one $x \in X$. Then there exists a convex cone $K \subset X$ with non-empty interior, such that A be maximal K-monotone.*

Proof. If for $x_0 \in X$ the inequality $\langle x_0, Ax_0\rangle > 0$ holds, then as A is continuous, there exists a convex neighbourhood V of x_0 such that $\langle x, Ax\rangle > 0$ for every $x \in V$. Now, it is clear that A is K-monotone with respect to the convex cone with non-empty interior : $K = \{tx;\ x \in V$ and $t \geqslant 0\}$ and we may apply Theorem 3.1.

Remark 3.1. If the convex cone K satisfies $K \cup (-K) = X$, then the K-monotonicity and the K-hemicontinuity are respectively equivalent to the monotonicity and the hemicontinuity and so, a well-known result [4, Lemma 2] about the maximal monotonicity of hemicontinuous monotone operators is derived from Theorem 3.1.

4. Local boundedness and continuity of K-monotone operators. An operator $A : X \to 2^{X^*}$ is said to be *locally bounded at* $x \in X$ if there exists a neighbourhood V of x such that the set $A(V) = \cup\{Ay;\ y \in D(A) \cap V\}$ is bounded in X^*.

THEOREM 4.1. *Let X be a real Banach space and let $K \subset X$ be a convex cone with non-empty interior. Then any K-monotone operator $A : X \to 2^{X^*}$ is locally bounded at the interior points of $D(A)$.*

Proof. Assume the existence of a point $x_0 \in \operatorname{int} D(A)$ in which A is not locally bounded. We may suppose that $x_0 = 0$ because the K-monotonicity is invariant under translations. Then there exist a sequence

$(x_n) \subset D(A)$ and a sequence (x_n^*), $x_n^* \in Ax_n$ such that $x_n \to 0$ and $\|x_n^*\| \to \infty$ as $n \to \infty$. Since K is a convex cone with non-empty interior, we can find a closed convex cone K_1 with non-empty interior such that $K_1 \subset \subset \operatorname{int} K \cup \{0\}$. We shall prove that for any $\rho > 0$ there exists a $z \in M_\rho = \bar{B}(0; 2\rho) \cap (X \setminus B(0; \rho)) \cap (K_1 \cup (-K_1))$ such that

$$\langle x_j - z, x_j^* \rangle \to -\infty \text{ as } j \to \infty, \tag{4.1}$$

for a certain subsequence (x_j) of (x_n) (where by $B(0; \rho)$ we have denoted the set $\{x \in X;\ \|x\| < \rho\}$). Assuming that this is not the case, we shall derive a contradiction. Let $\rho > 0$ such that for each $z \in M_\rho$ there exists a constant c_z for which the inequality $\langle x_n - z, x_n^* \rangle \geqslant c_z$ holds for all n. For any natural number k, the set $E_k = \{x \in M_\rho;\ \langle x_n - x, x_n^* \rangle \geqslant -k$ for all $n\}$ is closed and $M_\rho = \bigcup_{k=1}^{\infty} E_k$. Since X is complete and M_ρ is closed, we may use the Baire category theorem. Thus, there exist $y \in M_\rho$, $r > 0$ and k_0 such that $\bar{B}(y; r) \subset E_{k_0}$. Since $\langle x_n + y, x_n^* \rangle \geqslant c_{-y} = c$ and $\langle x_n - x, x_n^* \rangle \geqslant -k_0$ for any $x \in \bar{B}(y; r)$ and all n, we obtain $\langle 2x_n + y - x, x_n^* \rangle \geqslant c - k_0$ for $x \in \bar{B}(y; r)$ and all n. Now, we fix n_0 so that $\|x_n\| \leqslant r/4$ for all $n \geqslant n_0$. For $u \in \bar{B}(0; r/2)$ and $n \geqslant n_0$, we have $x = 2x_n + y - u \in \bar{B}(y; r)$ and consequently $\langle u, x_n^* \rangle \geqslant c - k_0$. Replacing u by $-u$, we obtain $-\langle u, x_n^* \rangle \geqslant c - k_0$. Therefore, $|\langle u, x_n^* \rangle| \leqslant k_0 - c$ for every $n \geqslant n_0$ and $u \in \bar{B}(0; r/2)$, which implies the boundedness of the sequence $(\|x_n^*\|)$, which is a contradiction. Thus, we have proved that for any $\rho > 0$ there exists a $z \in M_\rho$ such that relation (4.1) is satisfied for a certain subsequence (x_j) of (x_n). Now, since $x_0 = 0 \in \operatorname{int} D(A)$, we may choose the number $\rho > 0$ such that $\bar{B}(0; 2\rho) \subset D(A)$ and so, its corresponding element $z \in M_\rho$ belongs to $D(A)$. On the other hand, as $z \neq 0$ and $z \in K_1 \cup (-K_1)$, we have that $z \in \operatorname{int} K \cup (-\operatorname{int} K)$ and for n large enough, $x_n - z \in K \cup (-K)$. Using the K-monotonicity of A, we obtain $\langle x_j - z, x_j^* \rangle \geqslant \langle x_j - z, z^* \rangle$ for j large enough and for any $z^* \in Az$. Since the second term in the above inequality is bounded from below, we have arrived to a contradiction with (4.1). This proves the local boundedness of A at x_0 as claimed.

COROLLARY 4.1 ([6]). *If X is a real Banach space, then any monotone operator $A : X \to 2^{X^*}$ is locally bounded at the interior points of $D(A)$.*

Proof. Theorem 4.1 can be applied, where $K = X$.

COROLLARY 4.2. *Let X be a real Banach space and let $K \subset X$ be a convex cone with non-empty interior. Then, any (o)-monotone operator $A : X \to 2^{X^*}$ is locally bounded at the interior points of $D(A)$.*

Proof. Recall that each (o)-monotone operator is K-monotone and apply Theorem 4.1.

We shall also give a direct proof of Corollary 4.2. For this, let $x_0 \in \operatorname{int} D(A)$ and let $r > 0$ such that $B(x_0; r) \subset D(A)$. We consider [7, 3.1.3] $u \in \operatorname{int} K$ such that $B(0; r) = B(x_0; r) - x_0 \subset (-u + K) \cap (u - K)$ and we fix $\mu > 0$ for which $x_0 - \mu u$, $x_0 + \mu u \in D(A)$. Then, $B(0; \mu r) = B(x_0; \mu r) - x_0 \subset (-\mu u + K) \cap (\mu u - K)$. Therefore, $B(x_0; \mu r) \subset \subset (x_0 - \mu u + K) \cap (x_0 + \mu u - K)$. Next, we fix $f \in A(x_0 - \mu u)$ and $g \in A(x_0 + \mu u)$. The operator A being (o)-monotone, we have $\langle y, f \rangle \leqslant \langle y, x^* \rangle \leqslant \langle y, g \rangle$ for every $x \in B(x_0; \mu r)$, $x^* \in Ax$ and $y \in K$. Now, let z be any element of X. Since z can be written as $y_1 - y_2$ with $y_1, y_2 \in K$, we get $\langle y_1, f \rangle - \langle y_2, g \rangle \leqslant \langle z, x^* \rangle \leqslant \langle y_1, g \rangle - \langle y_2, f \rangle$ for all $x \in B(x_0; \mu r)$ and $x^* \in Ax$. Using the uniform boundedness theorem, the last inequalities imply that $A(B(x_0; \mu r))$ is bounded in X^*. Thus, we have proved the local boundedness of A at x_0 as claimed.

An operator $A : D(A) \to X^*$, $D(A) \subset X$, is said to be *demicontinuous at x_0* if $Ax_n \to Ax_0$ (as $n \to \infty$) weakly in X^*, for any sequence $(x_n) \subset D(A)$ strongly convergent to x_0 in X.

THEOREM 4.2. *Let X be a reflexive Banach space and let $K \subset X$ be a convex cone with non-empty interior. Let $A : D(A) \to X^*$, $D(A) \subset X$, be a K-monotone operator and let $x_0 \in \operatorname{int} D(A)$. If the operator A is K-hemicontinuous at x_0, then it is even demicontinuous at x_0.*

Proof. Let $(x_n) \subset \operatorname{int} D(A)$ be such that $x_n \to x_0$ as $n \to \infty$. According to Theorem 4.1, the sequence (Ax_n) is bounded in X^* and, by the reflexivity of X, passing if necessary to a subsequence, we may assume that $Ax_n \to x_0^*$ weakly in X^* for $n \to \infty$. Let x be any element of $D(A)$ for which $x - x_0 \in \operatorname{int} K \cup (-\operatorname{int} K)$. Then for n large enough, $x - x_n \in K \cup (-K)$ and by the K-monotonicity of A, we have $\langle x - x_n, Ax - Ax_n \rangle \geqslant 0$. Passing to the limit, we obtain

$$\langle x - x_0, Ax - x_0^* \rangle \geqslant 0 \tag{4.2}$$

for every $x \in D(A)$ satisfying $x - x_0 \in \operatorname{int} K \cup (-\operatorname{int} K)$. Since $x_0 \in \operatorname{int} D(A)$, to any $u \in \operatorname{int} K \cup (-\operatorname{int} K)$ there corresponds a $t_u > 0$ such that $x_0 + tu \in \operatorname{int} D(A)$ for all t with $0 < t \leqslant t_u$. Taking $x = x_0 + tu$ in (4.2), we obtain that $\langle u, A(x_0 + tu) - x_0^* \rangle \geqslant 0$ for $0 < t \leqslant t_u$. Then, by the K-hemicontinuity of A at x_0, we have that $\langle u, Ax_0 - x_0^* \rangle \geqslant 0$ for every $u \in \operatorname{int} K \cup (-\operatorname{int} K)$. But since $\operatorname{int} K + (-\operatorname{int} K) = X$, it follows that this inequality holds for all $u \in X$. Therefore, $Ax_0 = x_0^*$, that is A is demicontinuous at x_0.

COROLLARY 4.3 ([9]). *Let X be a reflexive Banach space. Then, any monotone hemicontinuous operator $A : D(A) \to X^*$, $D(A) \subset X$, is demicontinuous on $\operatorname{int} D(A)$.*

Proof. Apply Theorem 4.2, where $K = X$.

COROLLARY 4.4. *Let X be a reflexive Banach space and let $K \subset X$ be a convex cone with non-empty interior. Then, any (o)-monotone hemicontinuous operator $A : D(A) \to X^*$, $D(A) \subset X$, is demicontinuous on $\operatorname{int} D(A)$.*

5. Surjectivity of K-monotone operators. Let X be a real linear normed space. An operator $A : X \to X^*$ is said to be *coercive with respect to the element $h \in X^*$* if there exists a number $r > 0$ such that $\|x\| \geqslant r$ implies that $\langle x, Ax - h \rangle > 0$. The operator A is called *coercive* if $\langle x, Ax \rangle / \|x\| \to \infty$ as $\|x\| \to \infty$.

We shall use the following proposition (see [12, 3.2.8]).

LEMMA 5.1. *Let X be a finite-dimensional Banach space. If $A : X \to X^*$ is a continuous operator which is coercive with respect to the element $h \in X^*$, then there exists at least one element $x \in X$ such that $h = Ax$.*

THEOREM 5.1. *Let X be a reflexive Banach space and let $K \subset X$ be a convex cone having non-empty interior with respect to the weak topology on X. If $A : X \to X^*$ is a K-hemicontinuous K-monotone operator which is coercive with respect to $h \in X^*$, then there exists at least one element $x \in X$ such that $h = Ax$.*

Proof. Denote by int K and w-int K, the interiors of K with respect to the strong topology and to the weak topology on X, respectively. Obviously, w-int $K \subset$ int K and since w-int $K \neq \emptyset$, we have w-int $K + (-w\text{-int } K) = X$.

Let A be coercive with respect to $h \in X^*$. Then the operator $Bx = Ax - h$ is K-hemicontinuous, K-monotone and coercive with respect to 0. Thus, if one considers the operator B instead of A we may assume that $h = 0$. Therefore, we have to prove that there exists an $x_0 \in X$ such that $Ax_0 = 0$.

Let $\mathscr{H}$ be the family of all finite-dimensional linear subspaces H of X for which $H \cap \text{int } K \neq \emptyset$, ordered by inclusion. It is obvious that if $H \in \mathscr{H}$, then the convex cone $K \cap H$ has non-empty interior in the topology induced on H. For each $H \in \mathscr{H}$, let J_H be the injection map of H into X and let J_H^* be the dual projection map (the surjection) of X^* onto H^*. We set $A_H = J_H^* A J_H$. Then, the operator $A_H : H \to H^*$ is $K \cap H$-monotone and $K \cap H$-hemicontinuous and, by Theorem 4.2, it is continuous. Also, A_H is coercive with respect to 0. According to Lemma 5.1, there exists an $x_H \in H$ such that $A_H x_H = 0$. Consequently $\langle x_H, Ax_H\rangle = \langle x_H, A_H x_H\rangle = 0$. Then since A is coercive with respect to 0, it follows that there exists a constant C independent of H such that $\|x_H\| \leqslant C$ for every $H \in \mathscr{H}$.

Now, for any $H_0 \in \mathscr{H}$ consider the subset $V_{H_0} = \{x_H;\ H_0 \subset H\}$ of $\bar{B}(0; C)$. Then the family $\{V_{H_0};\ H_0 \in \mathscr{H}\}$ has the finite-intersection property. Indeed, if $H_1, H_2 \in \mathscr{H}$ and we denote by $H_0 = H_1 \cup H_2$, then $V_{H_0} \subset V_{H_1} \cap V_{H_2}$. Since X is reflexive, the ball $\bar{B}(0; C)$ is weakly compact and thus there exists an element $x_0 \in X$ which belongs to the weak closure of each set V_{H_0} with $H_0 \in \mathscr{H}$.

Let x be an arbitrary element of X such that $x - x_0 \in w$-int $K \cup (-w\text{-int } K)$ and let $H_0 \in \mathscr{H}$ be such that $x \in H_0$. Since x_0 belongs to the weak closure of V_{H_0}, there exists a sequence $(x_n) \subset V_{H_0}$ such that $x_n \to x_0$ (for $n \to \infty$) weakly in X. We may assume that $x - x_n \in w$-int $K \cup (-w\text{-int } K) \subset K \cup (-K)$ for all n. Then, by the K-monotonicity of A, we have that

$$(5.1) \qquad \langle x - x_n, Ax\rangle \geqslant \langle x - x_n, Ax_n\rangle$$

for all n. On the other hand, by $x_n \in V_{H_0}$ it follows that there exists an $H_n \in \mathscr{H}$ such that $H_0 \subset H_n$, $x_n \in H_n$ and $A_{H_n} x_n = 0$. Then, $x - x_n \in H_n$ and we may write that $\langle x - x_n, Ax_n\rangle = \langle x - x_n, A_{H_n} x_n\rangle = 0$. Thus, by (5.1) we find that $\langle x - x_n, Ax\rangle \geqslant 0$ for all n. Passing to the limit as $n \to \infty$, we obtain that $\langle x - x_0, Ax\rangle \geqslant 0$ for every $x \in X$ satisfying $x - x_0 \in w$-int $K \cup (-w\text{-int } K)$. Now, observe that A satisfies all the assumptions of Lemma 3.1, where the convex cone $K' = w$–int $K \cup \{0\}$ having non-empty strong interior, stands for K. Applying Lemma 3.1, we obtain $Ax_0 = 0$, which completes the proof.

If we take $K = X$, then Theorem 5.1 implies the following well-known result:

COROLLARY 5.1 ([2], [11]). *If X is a reflexive Banach space and $A : X \to X^*$ is a coercive hemicontinuous monotone operator, then A is surjective.*

It is natural to ask if Theorem 5.1 is applicable to (o)-monotone operators. The answer is negative for infinite-dimensional spaces X, as follows from:

Remark 5.1. Let X be an infinite-dimensional linear normed space and let $K \subset X$ be a convex cone having non-empty interior with respect to the weak topology on X. Let $h \in X^*$. Then, an (o)-monotone operator $A : X \to X^*$ which is coercive with respect to h, does not exist.

To prove this, let us assume the existence of such an operator. We shall derive a contradiction. Let $x_0 \in w$-int K. Then, there exist $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$ such that

$$V = \{x \in X;\ |\langle x - x_0, x_i^*\rangle| < \varepsilon,\ i = 1, 2, \ldots, n\} \subset K.$$

Consider the convex cone $K_1 \subset K$, $K_1 = \{\lambda x;\ x \in V,\ \lambda \geqslant 0\}$ and denote $S = \bigcap_{i=1}^{n} \ker x_i^*$. Is is clear that $K^* \subset K_1^*$ and $\lambda x_0 + S \subset K_1$ for every $\lambda > 0$. Thus, if $x^* \in K_1^*$, then $\langle \lambda x_0 + x, x^*\rangle \geqslant 0$ for all $x \in S$ and $\lambda > 0$. Passing to limit as $\lambda \downarrow 0$, we obtain that $\langle x, x^*\rangle \geqslant 0$ for all $x \in S$. It follows that $\langle x, x^*\rangle = 0$ for all $x \in S$. Hence, $S \subset \ker x^*$ for every $x^* \in K_1^*$. Since X is an infinite-dimensional linear space, one has $S \neq \{0\}$. Now, we fix any $y \in S$, $y \neq 0$. For $\lambda > 0$, we have $\lambda x_0 - \lambda y \in K_1 \subset K$ and using the (o)-monotonicity of A, we obtain $A(\lambda x_0) - A(\lambda y) \in K^* \subset K_1^*$. Hence, there is $x_\lambda^* \in K_1^*$ such that $A(\lambda y) = A(\lambda x_0) - x_\lambda^*$. On the other hand, by $\lambda x_0 - 0 \in K$ we have $A(\lambda x_0) - A(0) \in K_1^*$, that is $A(\lambda x_0) = A(0) + y_\lambda^*$, where $y_\lambda^* \in K_1^*$. Therefore, $A(\lambda y) = A(0) + y_\lambda^* - x_\lambda^*$ for all $\lambda > 0$. Since $y \in S \subset \ker y_\lambda^* \cap \ker x_\lambda^*$, we get $\langle y, A(\lambda y) - h\rangle = \langle y, A(0) - h\rangle$ for all $\lambda > 0$. The operator A being coercive with respect to h, the inequality $\langle \lambda y, A(\lambda y) - h\rangle > 0$ must hold for $\lambda > 0$ large enough. Hence, $\langle y, A(0) - h\rangle > 0$ for any $y \in S$, $y \neq 0$ which is a contradiction because the codimension of S is finite while the dimension of ker $(A(0)\text{-}h)$ is infinite.

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High School of Informatics
Calea Turzii 140–142
3400 – Cluj-Napoca
Romania