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PEAK SETS, PROPER FACES AND BOUNDARIES

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1. In [1] J. W. Bunce and W. R. Zame have studied the peak sets the proper faces and the boundaries of a linear subspace H of $C(X)$ by using the dual H'; they have proved that a certain subset of the Choquet boundary of H is also a boundary for .

Without using H', we shall study similar problems; we shall prove that a corresponding subset of $\mathrm{Min}(H)$ — the boundary introduced by B. Fuchssteiner in [2] — is also a boundary for H.

2. Let X be a compact Hausdorff space and let H be a linear subspace of $C(X)$ which separates X and contains the constant functions. Let us denote $H^+ = \{h \in H : h \geqslant 0\}$. A non-empty subset $K \subset X$ is called a *peak set* if there exists an $h \in H^+$ such that $K = \{x \in X : h(x) = 0\}$. A non-empty set which is an intersection of peak sets is called a *generalized peak set*.

Let us denote by V the family of all generalized peak sets and by W the family of all minimal generalized peak sets. An application of Zorn's lemma shows that every $K \in V$ contains a $K' \in W$. Let

$$P(H) = \{x \in X : \{x\} \text{ is a peak set}\}$$

$$T(H) = \{x \in X : \{x\} \in V\}$$

$$S(H) = \cup \{K : K \in W\}.$$

Then $P(H) \subset T(H) \subset S(H)$.

A non-empty convex subset E of H^+ is called a *face* of H^+ if $0 < a < 1$, f, $g \in H^+$, $(1-a) f + ag \in E$ implies f, $g \in E$.

A non-empty subset E of H^+ is a face iff

(i) $E + E \subset E$, $aE \subset E$ for all $a \in R_+$, and

(ii) $f \in E$, $g \in H$, $0 \leqslant g \leqslant f$ implies $g \in E$ (see [1]).

A face E of H^+ is said to be *proper* if $E \neq H^+$; this is equivalent to $1 \notin E$.

Let us denote by F the family of all proper faces of H^+ and by M the family of all maximal proper faces; an application of Zorn's lemma shows that every proper face is contained in a maximal proper face.

If $E \subset H^+$, $K \subset X$, let

$$E^\perp = \{x \in X : f(x) = 0 \text{ for all } f \in E\}.$$

$$K^\perp = \{f \in H^+ : f(x) = 0 \text{ for all } x \in K\}.$$

It is easy to prove that:

(1) $K^\perp \in F$ and $K^{\perp\perp} = K$ for all $K \in V$.

(2) $E^\perp \in V$ and $E \subset E^{\perp\perp}$ for all $E \in F$.

(3) $K^\perp \in M$ for all $K \in W$.

(4) $E^\perp \in W$ and $E = E^{\perp\perp}$ for all $E \in M$.

In particular, we obtain

PROPOSITION 1 (see also [1]). *The map $K \to K^\perp$ is a bijection between W and M. The inverse of this map is $E \to E^\perp$.*

COROLLARY 1. *A point $x \in X$ is in $S(H)$ iff $\{x\}^\perp \in M$.*

Proof. Let $x \in K \in W$. Then $\{x\}^\perp \in F$ and $\{x\}^\perp \supset K^\perp \in M$. This yields $\{x\}^\perp = K^\perp \in M$.

Conversely, let $\{x\}^\perp \in M$. Then $x \in \{x\}^{\perp\perp} \in W$, hence $x \in S(H)$.

3. A set $M \subset X$ is called a *boundary* for H if for each $h \in H$ there exists an $x \in M$ such that $h(x) = \min h(X)$.

Let $\mathrm{Ch}(H)$ be the Choquet boundary of H. $\mathrm{Ch}(H)$ is a boundary for H and $\mathrm{cl}(\mathrm{Ch}(H))$ is contained in every closed boundary for H.

Let $h \in H$ and let K be a compact subset of X. Denote $h[K] = \{x \in K : h(x) = \min h(K)\}$.

Let ρ be a well-ordering relation in H and let f be the first element of H with respect to ρ. Let us define inductively

$$X_f = f[X];$$

$$X_h = h[\cap \{X_g : g \rho h, g \neq h\}] \text{ for all } h \in H, \quad h \neq f.$$

Denote $X_\rho = \cap \{X_h : h \in H\}$, $\mathrm{Min}\,(H) = \cup\{X_\rho :\ \rho$ is a well-ordering relation in $H\}$.

Then $\mathrm{Min}\,(H)$ is a boundary for H and $\mathrm{Min}(H) \subset \mathrm{Ch}(H)$ (see [2], [3], [4]).

Let $M(H) = \mathrm{Ch}(H) \cap S(H)$, $N(H) = \mathrm{Min}(H) \cap S(H)$.

LEMMA 1. *If $K \in V$ then* $\mathrm{Min}(H|_K) \subset \mathrm{Min}\,(H)$.

Proof. Let $G \subset H^+$ be such that $K = \{x \in X : g(x) = 0$ for all $g \in G\}$. Let r be a well-ordering relation in $H|_K$ and let ρ be the well-ordering relation induced by r in $H|_K \setminus \{0\}$.

Let $\varphi \in H|_K$ and $H_\varphi = \{h \in H : h|_K = \varphi\}$. In every set H_φ with $\varphi \neq 0$, let us consider a well-ordering relation r_φ. We have $G \subset H_0$; in H_0 let us consider a well-ordering relation r_0 such that $g\, r_0\, h$ for all $g \in G$ and all $h \in H_0 \setminus G$.

Let $f, g \in H$. We put $f \sigma g$ if

a) $f \in H_0$, $g \in H_\varphi$, $\varphi \neq 0$; or

b) $f \in H_\varphi$, $g \in H_\psi$, $\varphi, \psi \in H|_K \setminus \{0\}$, $\varphi \rho \psi$, $\varphi \neq \psi$; or

c) $f, g \in H_\varphi$, $\varphi \in H|_K$, $f r_\varphi g$.

Then σ is a well-ordering relation in H. With respect to σ, we have $\cap \{X_g : g \in H_0\} = K$ and, consequently, $X_\sigma = K_\rho = K_r$. This implies that $\mathrm{Min}(H|_K) \subset \mathrm{Min}(H)$.

THEOREM 1. *$N(H)$ is a boundary for H.*

Proof. Let $h \in H^+$ such that $A = \{x \in X : h(x) = 0\}$ is non-empty. Since $A \in V$, there exists a $K \in W$ with $K \subset A$. Let $x \in \mathrm{Min}\,(H|_K)$. By Lemma 1, $x \in \mathrm{Min}(H)$. From $x \in K \in W$, it follows that $x \in S(H)$, hence $x \in N(H)$. But $x \in A$, which implies that $h(x) = 0 = \min h(X)$.

COROLLARY 2 (see also [1]). *$M(H)$ is a boundary for H. We have*

(5) $$P(H) \subset T(H) \subset N(H) \subset M(H) \subset \mathrm{Ch}(H)$$

(6) $$N(H) \subset \mathrm{Min}\,(H) \subset \mathrm{Ch}(H)$$

(7) $$\mathrm{cl}(N(H)) = \mathrm{cl}(M(H)) = \mathrm{cl}(\mathrm{Min}(H)) = \mathrm{cl}(\mathrm{Ch}(H)).$$

Proof. It remains to prove only that $T(H) \subset N(H)$. Let $x \in T(H)$; denote $K = \{x\}$. We have $K \in V$ and $x \in \mathrm{Min}\,(H|_K)$. According to Lemma 1, $x \in \mathrm{Min}(H)$. Since $T(H) \subset S(H)$, we deduce that $x \in N(H)$.

COROLLARY 3. *If max $(h, 0) \in H$ for each $h \in H$, then*

$$T(H) = N(H) = M(H) = \mathrm{Min}(H).$$

Proof. If max $(h, 0) \in H$ for each $h \in H$, then $\mathrm{Min}(H) \subset T(H)$. According to Corollary 2, $T(H) = N(H) = \mathrm{Min}(H) \subset M(H)$. Let $x \in M(H)$. Then $x \in \mathrm{Ch}(H)$ and $x \in K \in W$. Let $y \in K$, $y \neq x$. Let U be an open neighborhood of x such that $y \notin U$. Since $x \in \mathrm{Ch}(H)$, there exists an $f \in H$ such that $f(x) < 1/2$ and $f > 1$ on $X \setminus U$. Let $h = \max(f - f(x), 0)$. We have $h \in H$, $h \geqslant 0$, $h(x) = 0$, $h(y) > 1/2$. If we denote $K' = K \cap \{z \in X : h(z) = 0\}$, then $x \in K' \in V$ and $y \notin K'$, i.e., K' is a proper subset of K. This contradicts the minimality of K. Hence, $K = \{x\}$ and so $x \in T(H)$.

REMARK 1. Let $H = \{h \in C[0,1]$: there exists an $a > 0$ (depending on h) such that h is constant on $[0, a]\}$. Then $P(H) = (0,1]$, $T(H) = [0,1]$.

It is not difficult to prove that if H is finite-dimensional, then every generalized peak set is a peak set; in particular, $P(H) = T(H)$.

REMARK 2. Examples in which the inclusion $\mathrm{Min}(H) \subset \mathrm{Ch}(H)$ is strict can be found in refs. [3], [4], [5]. An example of a linear subspace H for which the inclusions $T(H) \subset N(H) \subset \mathrm{Min}(H)$ and $M(H) \subset \mathrm{Ch}(H)$ are strict can be found in ref. [1], p. 228.

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