MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

**L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION**
**Tome 16, N° 1, 1987, pp. 81–85**

# GENERALIZED DOUBLE SEQUENCES

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**1. Introduction.** In [5], D. M. E. FOSTER and G. M. PHILLIPS defined axiomatically the real, symmetric means. In [11], we have dispensed with symmetry and in [12] tried to transpose the definition to the complex case However, I think that the natural frame to develop the axiomatic study of means, as well as its application to double sequences, is that of lattices. That is what I shall do in the sequel.

**2. Preliminary definitions.** Let $X$ be a lattice, that is a partially ordered set, any two of whose elements $x$ and $y$ have a greatest lower bound (g.l.b.) $x \wedge y$ and a least upper bound (l.u.b.) $x \vee y$. For example, a direct product $X_1 \times \ldots \times X_p = X$ of $p$-totally ordered sets (or chains) $X_j$, partially ordered by the rule $(x^1, \ldots, x^p) \leqslant (y^1, \ldots, y^p)$, if and only if $x^j \leqslant y^j$ for $j = 1, \ldots, p$, is a lattice. We shall call it a $p$-product lattice and, of course, $x \wedge y = (x^1 \wedge y^1, \ldots, x^p \wedge y^p)$ and $x \vee y = (x^1 \vee y^1, \ldots, x^p \vee y^p)$, where, in fact, $x^j \wedge y^j = \min\{x^j, y^j\}$ and $x^j \vee y^j = \max\{x^j, y^j\}$.

We suppose that the lattice $X$ is relatively $\sigma$-complete (see [4]) that is, every bounded, monotonous sequence $(x_n)_{n \geqslant 0}$ has a g.l.b. $\bigwedge_{n \geqslant 0} x_n$ and a l.u.b. $\bigvee_{n \geqslant 0} x_n$. If the sequence $(x_n)_{n \geqslant 0}$ is increasing and its g.l.b. is $x$, it is natural to consider that it converges to $x$ and we write, as usual, $x_n \nearrow x$. Analogously, if $(y_n)_{n \geqslant 0}$ is decreasing and its l.u.b. is $y$, we write $y_n \searrow y$.

If $X = X_1 \times \ldots \times X_p$ is a $p$-product lattice and $(x_n)_{n \geqslant 0}$ is a sequence in $X$ such that $x_n = (x_n^1, \ldots, x_n^p)$ and every $(x_n^j)_{n \geqslant 0}$ is monotonous (for $j = 1, \ldots, p$), then $x_n^j \nearrow x^j$ or $x_n^j \searrow x^j$ and we write $x_n \to x$, where $x = (x^1, \ldots, x^p)$. For a general lattice, one can consider the convergence with respect to the order (see [4]). A function $m : X \times X \to X$ is called continuous if $x_n \to x$ and $y_n \to y$ implies $m(x_n, y_n) \to m(x, y)$.

**3. Means.** We shall consider the following class of means: a continuous function $m : X \times X \to X$ belongs to the class $M$ of means if it has the following properties:

$$x \wedge y \leqslant m(x, y) \leqslant x \vee y \tag{1}$$

$$m(x, y) = x \Rightarrow x = y. \tag{2}$$

A general example of mean is :

$$m(x, y) = f^{-1}\left[\frac{f(x)\cdot g(x) + f(y)\cdot h(y)}{g(x) + h(y)}\right],$$

where $f, f^{-1}, g, h : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous. Most means that we know are of this form (see, for instance, [5] and [10]). So, for $f(x) = x^p$, $p \neq 0$ and $g(x) = h(x)$, we have a mean from [5]. Taking also $f(x) = x^p$, $g(x) = q$ and $h(y) = 1 - q$, with $q \in (0,1)$, we have :

$$m_{p,q}(x, y) = [q\cdot x^p + (1-q)\cdot y^p]^{1/p}$$

that is, a (non-symmetric) Minkowski mean. We denote :

$$m_{1,q} = a_q,\ m_{0,q} = g_q \text{ and } m_{1,q} = h_q,$$

the weighted arithmetic, geometric and harmonic means, respectively.

In what follows, we need to compare the means. Two means $m$ and $m'$ are comparable if $m \leqslant m'$ or $m' \leqslant m$, where $m \leqslant m'$ means that $m(x, y) \leqslant m'(x, y)$ for any $x, y \in X$. Some examples of comparable means are given in ref. [10]. Of course, for any $q \in (0, 1)$: $a_q \geqslant g_q \geqslant h_q$.

Because two weighted arithmetic means are not comparable, we have introduced in [11], the following notion : the means $m$ and $m'$ are weakly comparable if $m \prec m'$ or $m' \prec m$, where $m \prec m'$ means that $x < y$ implies $m(x, y) < m'(x, y)$ and $x > y$ implies $m(x,y) > m'(x,y)$. For example, on $\mathbb{R}_+$ we have for $p < q$ : $a_p \prec a_q$, $g_q \prec g_p$ and $h_q \prec h_p$.

If $X$ is a $p$-product lattice, one can consider also other relations or they may be deduced by a change of the definition of the order on $X : (x^1, \ldots, x^p) \leqslant_J (y^1, \ldots, y^p)$ if $x^j \leqslant y^j$ for $j \in J \subseteq \{1, \ldots, p\}$ and $x^j \geqslant y^j$ for $j \notin J$.

**4. Double sequences.** Given two means $m$ and $m'$ from $M$ and $x_0$, $y_0$ from $X$, one can define a double sequence $(x_n, y_n)_{n\geqslant 0}$ in two ways :

(3) $$x_{n+1} = m(x_n, y_n),\quad y_{n+1} = m'(x_{n+e}, y_n),\quad n \geqslant 0,$$

and choosing $e = 0$ or $e = 1$. For historical reasons, if $e = 1$, the double sequence is called Archimedean and if $e = 0$, it is called Gaussian. Namely, Archimedes' procedure for estimating $\pi$ may be put in the form (3) with $X = \mathbb{R}_+$, $e = 1$, $m = h_{1/2}$ and $m' = g_{1/2}$, while the famous arithmetic-geometric mean of Gauss corresponds to $X = \mathbb{R}_+$, $e = 0$, $m = a_{1/2}$ and $m' = g_{1/2}$.

THEOREM 1. *If $(x_n, y_n)_{n\geqslant 0}$ is an Archimedean double sequence in a relatively σ-complete p-product lattice X, then sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ are convergent and have a common limit $A(m, m'; x_0, y_0)$.*

*Proof.* Let $X = X_1 \times \ldots \times X_p$, $x_n = (x_n^1, \ldots, x_n^p)$ and $y_n = (y_n^1, \ldots, y_n^p)$. For $1 \leqslant j \leqslant p$, as $X_j$ is a chain, $x_0^j$ and $y_0^j$ are comparable. Suppose that $x_0^j < y_0^j$. From $x_0 \wedge y_0 \leqslant x_1 \leqslant x_0 \vee y_0$, we deduce that $x_0^j < x_1^j < y_0^j$, then from $x_1 \wedge y_0 \leqslant y_1 \leqslant x_1 \vee y_0$, we have $x_1^j < y_1^j < y_0^j$ and so on, obtaining finally :

(4) $$x_1^j < x_2^j < \ldots < x_n^j < \ldots < y_n^j < \ldots < y_2^j < y_1^j.$$



The lattice $X$ being relatively σ-complete, every chain $X_j$ is also so, and thus $x_n^j \nearrow x^j$ and $y_n^j \searrow y^j$. If $x_0^j > y_0^j$, all the inequalities from (4) must be reversed, hence $x_n^j \searrow x^j$ and $y_n^j \nearrow y^j$. Finally, $x_n \to x$ and $y_n \to y$, where $x = (x^1, \ldots, x^p)$ and $y = (y^1, \ldots, y^p)$. From (3), we have $x = m(x, y)$ and from (2), $x = y = A(m, m' : x_0, y_0)$.

THEOREM 2. *If $(x_n, y_n)_{n\geqslant 0}$ is a Gaussian double sequence defined by two (weak) comparable means on a relatively σ-complete p-product lattice X, then the sequences $(x_n)_{n\leqslant 0}$ and $(y_n)_{n\geqslant 0}$ are convergent and have a common limit $G(m, m'; x_0, y_0)$.*

*Proof.* Using the notations employed in Theorem 1, we get (4) if $m \leqslant m'$ and converse relations if $m \geqslant m'$, but the proof may be continued in the same way. The proof is similar if $m \prec m'$, but if $m' \prec m$, we get :

$$x_0^j < y_1^j < x_2^j < \ldots < y_2^j < x_1^j < y_0^j$$

or the opposite inequalities if $x_0^j > y_0^j$. Hence, the sequences $(x_{2n})_{n\geqslant 0}$ and $(y_{2n+1})_{n\geqslant 0}$ have the same limit $x$, while $(x_{2n+1})_{n\geqslant 0}$ and $(y_{2n})_{n\geqslant 0}$ converge to $y$. From $m'(x_{2n}, y_{2n}) = y_{2n+1}$, we get $m'(x, y) = x$, thus $x = y = G(m, m'; x_0, y_0)$.

We must remark that the limit is also known in some cases. So, for $X = \mathbb{R}_+$ and $m, m' \in \{a_{1/2}, g_{1/2}, h_{1/2}\}$ their values may be found in refs. [2], [6] and [8]. A conjecture of G. D. SONG on $G(a_p, g_p; x_0, y_0)$ may be found in ref. [7]. Also, some results for weighted means were given in ref. [11].

**5. Multiple sequences.** In [1], a triplet of sequences defined by three means is proposed for study. This may be generalized as follows : a continuous function $m : X^k \to X$ belongs to the class $M^k$ of means if it satisfies the properties :

$$x^1 \wedge \ldots \wedge x^k \leqslant m(x^1, \ldots, x^k) \leqslant x^1 \vee \ldots \vee x^k$$

$$m(x^1, \ldots, x^k) = x^1 \wedge \ldots \wedge x^k \Rightarrow x^1 = \ldots = x^k.$$

Starting from $k$ means $m_1, \ldots, m_k \in M^k$ and $k$ initial values $x_0^1, \ldots, x_0^k$, one can construct $k$-uple of sequences by :

$$x_{n+1}^j = m_j(x_{n+e_{1j}}^1, \ldots, x_{n+e_{kj}}^k),\quad n \geqslant 0,\ j = 0, \ldots, k,$$

where $e_{ij} \in \{0, 1\}$, $e_{ij} = 0$ if $i \geqslant j$. Putting $E = (e_{ij})_{1\leqslant i,j\leqslant k}$ if $e_{ij} = 1$ for $j > i$, one can prove, as in Theorem 1, the convergence of the sequences $(x_n^j)_{n\geqslant 0}$ for $j = 1, \ldots, k$ to the same limit $L(E; m_1, \ldots, m_k; x_0^1, \ldots, x_0^k)$. But, if $e_{ij} = 0$ for some $j > i$, we must add a comparison hypothesis on the means $m_1, \ldots, m_k$ to get the convergence.

**6. The complex case.** The study of complex double sequences was begun even by Gauss. Some of the difficulties which can occur in this case are pointed out in refs. [3] and [9]. For example, even the arithmetic mean does not give the monotony of the modulus. Another problem is that the means are no longer single-valued and so, starting from one pair of initial values, we get more double sequences. To avoid these difficulties, we can change the partial order used on $C$ and also choose only "right" values for means, as is indicated in ref. [3] and more generally in ref. [12].

So, two partial orders are usually given on $C$. Following the first one, $z \leqslant w$ if $|z| \leqslant |w|$ and $\arg z \leqslant \arg w$, while, upon the second, $z \leqslant w$ if $\mathrm{Re}\, z \leqslant \mathrm{Re}\, w$ and $\mathrm{Im}\, z \leqslant \mathrm{Im}\, w$. In both these oders, $C$ is a 2-product lattice. In what follows, we suppose on $C$ one of these partial orders.

A continuous function $m : C \times C \to C$ belongs to the class $M$ if :

$$m(z, w) = z \Rightarrow x = w.$$

The value of $m(z, w)$ is called right if :

$$z \wedge w \leqslant m(z, w) \leqslant z \vee w.$$

Two functions $m, m' \in M$ are comparable on the set $D \subseteq C \times C$ if their components are comparable on $D$, that is :

$$\arg m(z, w) \leqslant \arg m'(z, w), \ \forall\ (z, w) \in D$$

or

$$\arg m(z, w) \geqslant \arg m'(z, w), \ \forall\ (z, w) \in D$$

and

$$|m(z, w)| \leqslant |m'(z, w)|, \ \forall\ (z, w) \in D$$

or

$$|m(z, w)| \geqslant |m'(z, w)|, \ \forall\ (z, w) \in D,$$

respectively :

$$\mathrm{Re}\, m(z, w) \leqslant \mathrm{Re}\, m'(z, w), \ \forall\ (z, w) \in D$$

or

$$\mathrm{Re}\, m(z, w) \geqslant \mathrm{Re}\, m'(z, w), \ \forall\ (z, w) \in D$$

and

$$\mathrm{Im}\, m(z, w) \leqslant \mathrm{Im}\, m'(z, w), \ \forall\ (z, w) \in D$$

or

$$\mathrm{Im}\, m(z, w) \geqslant \mathrm{Im}\, m'(z, w) \ \forall\ (z, w) \in D.$$

Let $m, m' \in M$ and $e \in \{0,1\}$. For $z_0, w_0 \in C$, we define a pair of sequences by :

$$z_{n+1} = m(z_n, w_n),\ w_{n+1} = m'(z_{n+e}, w_n), \quad n \geqslant 0.$$

The pair of sequences is called good if the values of $z_{n+1}$ and $w_{n+1}$ are right for all but finitely many $n > 0$. If $e = 0$, we suppose also that for some $n_0 \geqslant 0$, $m$ and $m'$ are comparable on the set $D = \{(z_n, w_n) : n \geqslant n_0\}$. Similarly to theorems 1 and 2, one can prove :

THEOREM 3. *Any good pair of sequences converges to a common limit.*

There are some open questions. So, we do not known what happens with a pair of sequences if $X$ is not a $p$-product lattice, or if $m$ and $m'$ are not (weakly) comparable for $e = 0$, or if the pair is not good for $X = C$.



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Received 14.II.1987

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