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ON STRONG APPROXIMATION OF FOURIER SERIES

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1. Introduction. Leindler [I, Theorem C] proved that if

$$\left\| \sum_{n=1}^{\infty} n^{(r+\alpha)\delta-1} |S_n - f|^{\delta} \right\| < \infty \tag{1}$$

then the r th derivative $f^{(r)}$ of f belongs to the Lipschitz class $\mathrm{Lip}(\alpha)$, where

$$f = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{2}$$

$\delta > 0$, $0 < \alpha < 1$, r being a nonnegative integer, and S_n is the partial sum of the Fourier series of the 2π periodic continuous function $f(x)$.

Leindler also proved that under the same conditions, $f^{(r)}$ and $\bar{f}^{(r)}$ belong to the little class lip (α), where for the special value $\delta = 2$ and $0 < \alpha < 1$, he proved that both $f^{(r)}$ and $\bar{f}^{(r)}$ belong to $\mathrm{Lip}(\alpha)$ for any positive integer r, and for $\delta = 2, \alpha = 1$, $f^{(r)}$ and $\bar{f}^{(r)}$ belong to Lip(1).

The purpose of the present paper, among others, is to link those results of Leindler with theorems on the order of magnitude of the Fourier coefficients of f.

2. Definitions and Notation. In this work $L^p(T)$ denotes the L^p space of the 2π periodic functions on the circle group T. T^n denotes the n-dimensional torus group, $\| \cdot \|_p$ and $\| \cdot \|_\infty$ stand for the L^p and supremum norms, respectively.

For convenience, we shall be dealing with the complex form

$$f(x) = \sum_{|n|=0} c_n e^{-inx}$$

instead of (2), and for functions of several variables we write

$$f(x) = f(x_1, \ldots, x_m)$$
$$= \sum_{n_1} \cdots \sum_{n_m} C_{n_1, \ldots, n_m} e^{-i(x \cdot n)} .$$

DEFINITION 2.1. *Let $f(x)$ belong to $L^p(T)$. Then the Lipschitz class* Lip (α, p) *is the collection of those functions $f(x)$ in $L^p(T)$ such that*

$$\|f(x+h)-f(x)\|_p = 0\,(h^\alpha)$$
$$0 < \alpha < 1 \text{ as } h \to 0.$$

The Little Lipschitz class lip (α, p) can be obtained by replacing O with o.

If, instead of using the first difference, we employ the rth difference with step h of f,

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x+ih),$$

then we can write

$$\|\Delta_h^r f\|_p = 0(h^\alpha)$$

and

$$\|\Delta_h^r f\|_p = o(h^\alpha),$$

respectively.

3. Main Results. Now, we state and prove

THEOREM 3.1. *Let $f(x)$ be a 2π periodic continuous function such that* (1) *holds; then the Fourier coefficients c_n of f belong to the sequence space l^β for*

$$\frac{p}{p(r+\alpha+1)-1} < \beta \leqslant p' = \frac{p}{p-1}$$
$$1 < p \leqslant 2.$$

Proof. If (1) holds, then the rth derivative $f^{(r)}$ of f belongs to the Lipschitz class Lip(α), $0 < \alpha \leqslant 1$, since by Leindler [1] $f^{(r)} \in$ Lip (1), so it belongs to Lip(α). $0 < \alpha \leqslant 1$, and hence $f^{(r)}$ is contained in the wider Lipschitz class Lip (α, p), $1 < p \leqslant 2$ on the circle group. This means that

$$\|f^{(r)}(x+h) - f^{(r)}(x)\|_p = O(h^\alpha) \tag{3}$$
$$0 < \alpha \leqslant 1, \text{ as } h \to 0.$$

However, it was proved [2, Theorem 2.6 p. 28] that if $g(x) \in$ Lip(α, p) over the circle group T, then its Fourier coefficients c_n belong to the sequence space l^β if

$$\frac{p}{p+\alpha p-1} < \beta \leqslant p' = \frac{p}{p-1}$$

In fact, we proved that

$$\sum_{n=1}^{N} |C_n|^\beta = 0[N^{1-\beta-\alpha\beta+\frac{\beta}{p}}] \tag{4}$$

as $N \to \infty$.

Now, it is very well known that if the Fourier coefficient of f is C_n, then that of its rth derivative $f^{(r)}$ is equal to $n^r c_n$. Hence, applying (4) to the Fourier coefficients of $f^{(r)}$, we obtain

$$\sum_{n=1}^{N} |n^r c_n|^\beta = O([N^{1-\beta-\alpha\beta+\frac{\beta}{p}}]. \tag{5}$$

An appeal to a lemma on the partial sums of sequences [3, p. 101] shows that (5) is equivalent to

$$\sum_{n=N}^{\infty} |C_n|^\beta = O[N^{1-\beta-\alpha\beta-r\beta+\frac{\beta}{p}}] \tag{6}$$

and the right-hand side of (6) is bounded as $N \to \infty$ if

$$1 - \beta - \alpha\beta - r\beta + \frac{\beta}{p} \leqslant 0$$

or, equivalent ly, if

$$\frac{p}{p(r+\alpha+1)-1} < \beta \leqslant p'$$

and the proof is complete.

We remark first that if $r = 0$, the last result reduces to a theorem proved in [2, Theorem 2.6, p. 28]. We also add that the special choice $\delta = 2$ and $\alpha = 1$ enables us to prove theorem 3.1 for both $f^{(r)}$ and its conjugate $\tilde{f}^{(r)}$, whereas the restrictions $\delta = 2$, $0 < \alpha < 1$ enable us to prove the theorem for $\tilde{f}^{(r)}$ (See [1, theorem C and thereafter]), but we shall not follows this course here any further.

3.2 The special case $p = 2$. This particular choice of $p = 2$ indicates a degree of symetry in theorem 3.1. In [2, Theorem 2.17, p. 42] we proved

THEOREM A. *Let $f(x)$ belong to $L^2(T)$. Then the conditions*

$$\|f(x+h) - f(x)\|_2 = O(h^\alpha). \tag{7}$$
$$0 < \alpha < 1 \text{ as } h \to 0$$

and

$$\sum_{|n| \geqslant N}^{\infty} |C_n|^2 = O[N^{-2\alpha}] \tag{8}$$

as $N \to \infty$, are equivalent.

Applying this result to $f^{(r)}$, we can now prove

THEOREM 3.2. *Let the conditions of theorem* 3.1 *be satisfied with $p=2$. Then*

$$\sum_{|n| \geqslant N}^{\infty} |C_n|^2 = O[N^{-2(\alpha+r)}]$$

as $N \to \infty$.

Proof. In this case, we use Parseval's identity and obtain

$$\sum_{n=1}^{N} | \hat{f}^{(r)} |^2 = O[N^{-2\alpha}],$$

where $\hat{f}^{(r)} = n^r c_n$ and, hence,

$$\sum_{n=1}^{N} | n |^{2r} \, | c_n |^2 = O[N^{-2\alpha}],$$

which is equivalent to the desired estimate by applying Duren's Lemma [3. p. 101].

Note 3.3. The special situation $\delta = 2$ and $\alpha = 1$ is of no genuine value for theorem 3.2. This is because our original theorem [2, Theorem, 2.17, p. 42] is not valid for $\alpha = 1$.

We also add that $f^{(r)} \in \mathrm{Lip}(\alpha)$, $0 < \alpha < 1$ is equivalent to saying that **the** r**th difference of** f belongs to Lip (α) for $\alpha' = r + \alpha$, and hence one can **formulate theorems** 3.1 and 3.2 in terms of higher differences and **obtain exactly the same** results.

4. Functions on T^n. In this section, we generalize theorems 3.1 and 3.2 to functions of several variables. For simplicity, we sketch the results for functions on T^2.

Let $f(x, y)$ be a 2π periodic and continuous function in x, y, let $0 < \alpha_1, \alpha_2 < 1$, r_1, r_2 being positive integers. $S_{m,n}$ will stand for the partial sum of the double Fourier series of f. Then condition (1) in this case becomes

(9) $$\| \sum_{m} \sum_{n} m^{(r_1+\alpha_1)\delta-1} \, n^{(r_2+\alpha_2)\delta-1} \, | S_{m,n} - f |^{\delta} \| < \infty$$

and this would imply that the partial derivative of order r_1 in x is in $\mathrm{Lip}(\alpha_1)$ and the partial derivative of order r_2 in y $\dfrac{\partial^{r_2} f}{\partial y^{r_2}}$ belongs to $\mathrm{Lip}(\alpha_2)$.

With these modifications, the proofs of theorems 3.1 and 3.2 can be carried almost verbally. The conclusion of theorem 3.1, for example, well asserts that the Fourier coefficients $C_{m,n}$ of f belong to $l^{\beta'}$, where $\beta = \max(\beta_1, \beta_2)$ where

$$\frac{p}{p(r_1 + \alpha_1 + 1) - 1} < \beta_1 \leqslant p'$$

$$\frac{p}{p(r_2 + \alpha_2 + 1) - 1} < \beta_2 \leqslant p'$$

In other words $C_{m,n} \in l^{\beta}$ for

$$\frac{p}{p(r + \alpha + 1) - 1} < \beta \leqslant p'$$

for

$$(r + \alpha) = \min [(r_1 + \alpha_1), (r_2 + \alpha_2)].$$

The special case of $L^2(T^2)$, i.e. $p = 2$, leads, however, to the equivalence of the following estimates

$$\sum_{|m| > M} \sum_{|n| > N} | C_{m,n} |^2 = 0[M^{-2(r_1+\alpha_1)} N^{-2(r_2+\alpha_2)}]$$

$$\sum_{|m| > M} \sum_{|n| \leqslant N} | nC_{m,n} |^2 = O[M^{-2(r_1+\alpha_1)} N^{2-(r_2+\alpha_2)}]$$

$$\sum_{|m| \leqslant M} \sum_{|n| \geqslant N} | mC_{m,n} |^2 = O[M^{2-2(r_1+\alpha_1)} N^{-2(r_2+\alpha_2)}]$$

Finally, we remark that for functions on $L^p(T^n)$, the lines of thoughts are clear and the proofs are direct but it would be rather complicated to produce them here.

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