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ON THE ORDER OF SIMULTANEOUS APPROXIMATION OF BIVARIATE FUNCTIONS BY BERNSTEIN OPERATORS

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Abstract. We prove an inequality of the form

$$||D^{r,s} B_{m,n} \; f - D^{r,s} f|| \leqslant t(r, s)\, \omega\left(D^{r,s} f\,;\, \frac{1}{\sqrt{m-r}}, \frac{1}{\sqrt{n-s}}\right) +$$

$$+ \max\left\{\frac{r(r-1)}{m}, \frac{s(s-1)}{n}\right\} ||D^{r,s} f||,$$

where $m > r \geqslant 0$, $n > s \geqslant 0$ are integers, f is a real-valued function on $[0,1]^2 = [0,1] \times [0,1]$ such that $D^{r,s} f$ is continuous (here $D^{r,s} f$ is the differential operator of order (r, s), i.e. $D^{r,s} f = \dfrac{\partial^{r,s} f(x, y)}{\partial x^r \, \partial y^s}\Big)$, $B_{m,n}$ is the Bernstein operator of order (m, n), $\omega(g\,;\,.,.)$ is the first-order modulus of continuity of the bivariate continuous function g and t is a certain real-valued function on $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ (here, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $||.||$ is the supremum norm).

This inequality improves an inequality established in 1974 by Ion Badea [1]. Because $t(0, 0) = (1390 + 837\sqrt{6})/2916 \approx 1.1797746\ldots$, the above inequality extends on simultaneous approximation another inequality due to the first author [2].

1. Introduction. Let f be a real-valued function defined on $[0, 1]^2 = [0, 1] \times [0,1]$ and let m, n be two positive integers. Let $B_{m,n}$ be the Bernstein operator of order (m, n) given by

$$B_{m,n}(f;\, x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f(i/m,\, j/n) p_{m,i}(x) p_{n,j}(y),$$

where

$$p_{k,l} = \binom{k}{l} u^l (1-u)^{k-l}.$$

For every integer $r, s \geqslant 0$, we denote by $D^{r,s} f$ the differential operator of order (r, s), given by

$$D^{r,s} f(x, y) = \frac{\partial^{r+s} f(x, y)}{\partial x^r \, \partial y^s}.$$

and by $C^{r,s}[0,1]^2$ the linear space of real-valued functions on $[0,1]^2$ such that $D^{r,s}f$ is continuous.

From one of E. H. Kingsely's theorems [7], we know that the Bernstein operator has the property of simultaneous approximation, i.e. for every $r, s \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $f \in C^{r,s}[0,1]^2$, we have

$$\lim_{m,n\to\infty} D^{r,s}B_{m,n}f(x,y) = D^{r,s}f(x,y), \tag{1.1}$$

uniformly on $[0,1]^2$.

Quantitative versions for relation (1.1), involving the first-order modulus of continuity $\omega(D^{r,s}f;\cdot,\cdot)$, were given by many authors: firstly by D. D. Stancu [14] in 1960 and then by G. Moldovan and I. Rîp [11] in 1966 and I. Badea [1] in 1974.

The quantitative version of (1.1) from [1] became, for $r = s = 0$, the following inequality which was first proved in [2]:

$$\|f - B_{m,n}f\| \leqslant (2K-1)\,\omega\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right), \tag{1.2}$$

where $\|\cdot\|$ is the supremum norm and $K = (4306 + 837\sqrt{6})/5832 \approx 1.0898873\ldots$ is Sikkema's constant (we note that $2K - 1 = (1390 + 837\sqrt{6})/2916 \approx 1.1797746\ldots$) (see [13]).

Inequalities weaker than (1.2) were given by A. F. Ipatov [6] and G. Moldovan and I. Rîp [11]. For general estimations of this type, see also Moldovan [10].

The purpose of this paper is to improve the quantitative assertion from [1] such that for $r = s = 0$ we obtain again inequality (1.2).

Finally, we note that the univariate similar problem was considered by many authors; see e.g. Stancu [14], Moldovan [9], Badea [1], [3], Knoop and Pottinger [8] and the recent references of Gonska [5] and the authors of the present paper [4].

2. Preliminary results. We shall need some preliminary results in our consideration.

LEMMA 1 ([12]). *If $r, s \in \mathbb{N}$, then for every non-negative numbers $\delta_1, \delta_2, \lambda_1, \lambda_2$, we have*

$$\omega(D^{r,s}f; \lambda_1\delta_1, \lambda_2\delta_2) \leqslant \{1 + \max(]\lambda_1[,]\lambda_2[)\}\,\omega(D^{r,s}f; \delta_1, \delta_2).$$

In this paper, $]\lambda[$ is the greatest integer being smaller than λ.

LEMMA 2. *If $f \in C^{r,s}[0,1]^2$ and $m > r$, $n > s$, we have*

$$\|D^{r,s}f - D^{r,s}B_{m,n}f\| \leqslant \left(2 + \frac{\sqrt{r}}{2} + \frac{\sqrt{s}}{2}\right)\omega\left(D^{r,s}f; \frac{1}{\sqrt{m-r}}, \frac{1}{\sqrt{n-s}}\right) + M^{r,s}_{m,n}(f),$$

where $M^{r,s}_{m,n}(f) = \max\left\{\dfrac{r(r-1)}{m}, \dfrac{s(s-1)}{n}\right\}\|D^{r,s}f\|.$

Proof. It is known [14] that

$$D^{r,s}B_{m,n}f(x,y) = C(m,r)\,C(n,s)\sum_{i=0}^{m-r}\sum_{j=0}^{n-s} D^{r,s}f(\xi_i, \theta_j)\,p_{m-r,i}(x)\,p_{n-s,j}(y),$$

where $i/m < \xi_i < (i+r)/m$, $j/m < \theta_j < (j+s)/n$ and

$$C(u,v) = \prod_{k=1}^{v-1}\left(1 - \frac{k}{u}\right) \text{ if } v \geqslant 2 \text{ and } C(u,1) = 1.$$

Using this formula and the equality $\sum_{i=0}^{m-r} p_{m-r,i}(x) = 1$, we can write:

$$|D^{r,s}f(x,y) - D^{r,s}B_{m,n}f(x,y)| \leqslant \sum_{i=0}^{m-r}\sum_{j=0}^{n-s}|D^{r,s}f(x,y) - D^{r,s}f(\xi_i, \theta_j)|\,p_{m-r,i}(x)\,p_{n-s,j}(y) + \{1 - C(m,r)\,C(n,s)\}\,\|D^{r,s}f\|. \tag{2.1}$$

Firstly, we prove the relation

$$1 - C(m,r)\,C(n,s) \leqslant \max\left\{\frac{r(r-1)}{m}, \frac{s(s-1)}{n}\right\}. \tag{2.2}$$

Because $0 < C(m,r) \leqslant 1$, we have

$$1 - C(m,r)\,C(n,s) \leqslant \{1 + \min(C(m,r), C(n,s))\} \times \times \{1 - \min(C(m,r), C(n,s))\} \leqslant 2\{1 - \min(C(m,r), C(n,s))\}. \tag{2.3}$$

Moreover, using the inequality $1 - C(m,r) \leqslant \dfrac{r(r-1)}{2m}$ in (2.3) (see [14]), we get

$$1 - C(m,r)\,C(n,s) \leqslant \begin{cases} \dfrac{r(r-1)}{m}, & \text{if } C(m,r) \leqslant C(n,s) \\ \dfrac{s(s-1)}{n}, & \text{if } C(m,r) > C(n,s) \end{cases} \leqslant \tag{2.4}$$

$$\leqslant \max\left\{\frac{r(r-1)}{m}, \frac{s(s-1)}{n}\right\}.$$

Thus, relation (2.2) is true.

Now, we estimate the sum $S(x,y)$ given by

$$S(x,y) = \sum_{i=0}^{m-r}\sum_{j=0}^{n-s}|D^{r,s}f(x,y) - D^{r,s}f(\xi_i, \theta_j)|\,p_{m-r,i}(x)\,p_{n-s,j}(y). \tag{2.5}$$

Let $\delta_1, \delta_2 > 0$ be two real numbers which are independent of i and j. Thus, applying an inequality derived from Lemma 1, we find that

$$|D^{r,s}f(x,y) - D^{r,s}f(\xi_i, \theta_j)| \leqslant \omega(D^{r,s}f; |x - \xi_i|, |y - \theta_j|) \leqslant$$
$$\leqslant (1 + |x - \xi_i|\,\delta_1^{-1} + |y - \theta_j|\,\delta_2^{-1})\,\omega(D^{r,s}f; \delta_1, \delta_2). \tag{2.6}$$

Using inequality (2.6) above, we get

$$S(x, y) \leqslant \omega(D^{r,s}f;\, \delta_1, \delta_2)\left\{1 + \delta_1^{-1}\sum_{i=0}^{m-r}|x - \xi_i|p_{m-r,i}(x) + \right. \tag{2.7}$$

$$\left. + \delta_2^{-1}\sum_{j=0}^{n-s}|y - \theta_j|p_{n-s,j}(y)\right\}.$$

But in the proof of Lemma 4 of our previous paper [4], we established that

$$\sum_{i=0}^{m-r}|x - \xi_i|p_{m-r,i}(x) \leqslant \frac{1}{2\sqrt{m-r}} + \frac{r}{m}. \tag{2.8}$$

The following estimation follows from (2.7) and (2.8):

$$S(x, y) \leqslant \omega(D^{r,s}f;\, \delta_1, \delta_2)\left\{1 + \delta_1^{-1}\left(\frac{1}{2\sqrt{m-r}} + \frac{r}{m}\right) + \right. \tag{2.9}$$

$$\left. + \delta_2^{-1}\left(\frac{1}{2\sqrt{n-s}} + \frac{s}{n}\right)\right\}.$$

Choosing $\delta_1 = \dfrac{1}{\sqrt{m-r}}$, $\delta_2 = \dfrac{1}{\sqrt{n-s}}$ in (2.9), we obtain that

$$S(x, y) \leqslant \left(2 + \frac{r\sqrt{m-r}}{m} + \frac{s\sqrt{n-s}}{n}\right)\omega\left(D^{r,s}f;\, \frac{1}{\sqrt{m-r}},\, \frac{1}{\sqrt{n-s}}\right). \tag{210}$$

Keeping in mind that $\dfrac{r\sqrt{m-r}}{m} \leqslant \dfrac{\sqrt{r}}{2}$ ([4, Proof of Lemma 3]) and employing inequalities (2.1), (2.2) and (2.10), we have the desired result.

The last preliminary result is

LEMMA 3. *If $f \in C^{r,s}[0, 1]^2$ and $m > r$, $n > s$ we have*

$$\|D^{r,s}f - D^{r,s}B_{m,n}f\| \leqslant$$

$$\leqslant \{2K + \max\,(]\sqrt{r}/2\,[,]\,\sqrt{s}/2[)\}\omega\left(D^{r,s}f;\, \frac{1}{\sqrt{m-r}},\, \frac{1}{\sqrt{n-s}}\right) + M^{r,s}_{m,n}(f).$$

Remark. For $r = s = 0$, the above inequality becomes (1.2). Lemma 3 is a refinement of Theorem 2.3.5 from [1].

Proof of Lemma 3. The notations are similar to the notations used in the proof of Lemma 2.

We have, from (2.1), (2.2) and (2.5), that

$$|D^{r,s}f(x, y) - D^{r,s}B_{m,n}f(x, y)| \leqslant S(x, y) + M^{r,s}_{m,n}(f). \tag{2.11}$$

The sum $S(x, y)$ can be expressed as

$$S(x, y) = S_1(x, y) + S_2(x, y), \tag{2.12}$$

where

$$S_1(x, y) = \sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\left|D^{r,s}f(x, y) - D^{r,s}f\left(\frac{i}{m-r},\, \frac{j}{n-s}\right)\right|p_{m-r,i}(x)\,p_{n-s,j}(y)$$

and

$$S_2(x, y) =$$

$$= \sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\left|D^{r,s}f\left(\frac{i}{m-r},\, \frac{j}{n-s}\right) - D^{r,s}f(\xi_i,\, \theta_j)\right|p_{m-r,i}(x)\,p_{n-s,j}(y).$$

In [2], one of the authors proved that

$$S_1(x, y) \leqslant (2K - 1)\,\omega\left(D^{r,s}f;\, \frac{1}{\sqrt{m-r}},\, \frac{1}{\sqrt{n-s}}\right). \tag{2.13}$$

We now estimate $S_2(x, y)$.

In [3] and [4], we proved the following inequalities:

$\left|\dfrac{i}{m-r} - \xi_i\right| \leqslant \dfrac{r}{m}$, $\left|\dfrac{j}{n-s} - \theta_j\right| \leqslant \dfrac{S}{n}$. Because $\dfrac{q\sqrt{n-q}}{n} \leqslant \dfrac{\sqrt{q}}{2}$, we have $\dfrac{q}{n} \leqslant \dfrac{\sqrt{q}}{2\sqrt{n-q}}$, for $n > q$. Applying these facts and, moreover, using Lemma 1, we can write

$$S_2(x, y) \leqslant \omega\left(D^{r,s}f;\, \frac{\sqrt{r}}{2\sqrt{m-r}},\, \frac{\sqrt{s}}{2\sqrt{n-s}}\right) \leqslant \tag{2.14}$$

$$\leqslant \{1 + \max\,(]\sqrt{r}/2\,[,\,]\,\sqrt{s}/2[)\}\,\omega\left(D^{r,s}f;\, \frac{1}{\sqrt{m-r}},\, \frac{1}{\sqrt{n-s}}\right).$$

Adding inequalities (2.13) and (2.14), we get the desired result.

3. Main Theorem. We are now ready to prove the main result of this paper.

We shall consider the following real-valued function $t(r, s)$ defined on $\mathbb{N} \times \mathbb{N}$ by: $t(r, s) = 2 + \left(\dfrac{\sqrt{r}}{2} + \dfrac{\sqrt{s}}{2}\right)$, if $\min\,(r, s) = 0$ and there exists a q such that $q < \max\left(\dfrac{\sqrt{r}}{2}, \dfrac{\sqrt{s}}{2}\right) \leqslant q + 2K - 2$, and $t(r, s) = 2K + \max\,(]\sqrt{r}/2\,[,]\,\sqrt{s}/2[)$ elsewhere in $\mathbb{N}^2$, where $K \approx 1.08988733\ldots$ was defined in the first section.

Our main result is the following

THEOREM. *If $f \in C^{r,s}[0, 1]^2$ and $m > r$, $n > s$, we have*

$$\|D^{r,s}f - D^{r,s}B_{m,n}f\| \leqslant t(r, s)\,\omega\left(D^{r,s}f;\, \frac{1}{\sqrt{m-r}},\, \frac{1}{\sqrt{n-s}}\right) + M^{r,s}_{m,n}(f). \tag{3.1}$$

Proof. From Lemmas 2 and 3, we get that (3.1) is true with

$$g(r, s) = \min\left\{2 + \frac{\sqrt{r}}{2} + \frac{\sqrt{s}}{2},\ \ 2K + \max\left(]\sqrt{r}/2[,\ ]\sqrt{s}/2[\right)\right\}$$

instead of $t(r, s)$.

It is thus sufficient to prove that $t(r, s) = g(r, s)$, i.e.

$$2 + \frac{\sqrt{r}}{2} + \frac{\sqrt{s}}{2} \leqslant 2K + \max\left(]\sqrt{r}/2[,\ ]\sqrt{s}/2[\right) \tag{3.2}$$

if and only if $\min(r, s) = 0$ and there exists a $q \in \mathbb{N}$ such that

$$q < \max\left(\frac{\sqrt{r}}{2}, \frac{\sqrt{s}}{2}\right) \leqslant q + 2K - 2 \approx q + 0.17977466 \ldots .$$

Without loss of generality, we may assume that $\max(r, s) = s$.

First, we consider the case $s = 4q^2$, where $q \in \mathbb{N}$, *i.e.*

$$\max\left(\frac{\sqrt{r}}{2}, \frac{\sqrt{s}}{2}\right) = q.$$

In this case, we have $]\sqrt{s}/2[\, = \,]q[\, = q - 1$.

In this situation, (3.2) is equivalent to

$$2 + \frac{\sqrt{r}}{2} + q \leqslant 2K + q - 1. \tag{3.3}$$

or to

$$\frac{\sqrt{r}}{2} \leqslant 2K - 3. \tag{3.4}$$

But $2K - 3 < 0$ and, thus, (3.4) is not true.

Finally, we analyse (3.2) if there exists a $q \in \mathbb{N}$ such that

$$q < \max\left(\frac{\sqrt{r}}{2}, \frac{\sqrt{s}}{2}\right) = \frac{\sqrt{s}}{2} < q + 1.$$

In this case, $]\sqrt{s}/2[\, = [\sqrt{s}/2] = q$, where, as usual, $[\cdot]$ is the integral part.

In this situation, (3.2) is equivalent to

$$2 + \frac{\sqrt{r}}{2} + \frac{\sqrt{s}}{2} \leqslant 2K + q \tag{3.5}$$

or to

$$\frac{\sqrt{r}}{2} \leqslant 2K - 2 + q - \frac{\sqrt{s}}{2}. \tag{3.6}$$

We suppose that (3.6) is true. Because $q < \frac{\sqrt{s}}{2}$, we have $\frac{\sqrt{r}}{2} < 2K - 2 \approx 0.17977466 \ldots$; hence, $r = 0$.

In this situation, (3.6) becomes

$$\frac{\sqrt{s}}{2} \leqslant 2K - 2 + q. \tag{3.7}$$

Thus, we have proved that (3.2) is true if and only if $r = 0$ and there exists a positive integer q such that $q < \frac{\sqrt{s}}{2} < q + 2K - 2$.

The proof of the main Theorem is now complete.

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