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REPRESENTATION OF CONTINUOUS LINEAR FUNCTIONALS ON SMOOTH REFLEXIVE BANACH SPACES

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Abstract. The main purpose of this paper is to give some representation theorems for the continuous linear functionals on smooth reflexive Banach spaces by use of the semi-inner-product in the sense of Lumer or Tapia [4], [7].

Introduction. DEFINITION 1 ([4], [1] p. 386). Let X be a real or complex linear space. A mapping $(.,.)_L : X \times X \to \mathbb{K}$ is called a semi-inner-product in the sense of Lumer (L-semi-inner-product), if the following conditions are satisfied:

(i) $$(x+y, z)_L = (x, z)_L + (y, z)_L; \quad x, y, z \in X;$$

(i) $$(\lambda x, y)_L = \lambda(x, y)_L; \quad \lambda \in \mathbb{K}, \quad x, y \in X;$$

(iii) $$(x, x)_L > 0 \text{ if } x \neq 0;$$

(iv) $$|(x, y)_L|^2 \leqslant (x, x)_L (y, y)_L; \quad x, y \in X;$$

(v) $$(x, \lambda y)_L = \bar{\lambda}(x, y)_L; \quad \lambda \in \mathbb{K}, x, y \in X.$$

We note that the mapping $X \ni x \overset{\|\cdot\|}{\mapsto} (x, x)^{1/2} \in \mathbb{R}_+$ is a norm on X, and the functional given by $X \ni x \overset{f_y}{\mapsto} (x, y)_L \in \mathbb{K}$ is a continuous linear functional on the normed linear space $(X, \|\cdot\|)$ and $\|f_y\| = \|y\|$.

THEOREM A ([6], [1] p. 386). *Let $(X, \|.\|)$ be a normed linear space. Then every L-semi-inner-product which generates the norm is given by*

(1) $$(x, y)_L = \langle \widetilde{J}(y), x \rangle \quad x, \ y \in X,$$

where $\widetilde{J}$ is section of duality mapping on X.

COROLLARY ([1] p. 387). *Let $(X, \|\cdot\|)$ be a normed linear space. Then (i) there exists a unique L-semi-inner-product which generates the norm $\|\cdot\|$ iff $(X, \|\cdot\|)$ is a smooth normed linear space;*
(ii) an L-semi-inner-product on X which generates the norm $\|\cdot\|$ is a scalar product iff $(X, \|\cdot\|)$ is a prehilbertian space.

Further on, by using the notion of the continuous *L-semi-inner-product*, i.e. an *L-semi-inner-product* which verifies the condition

(C) $$\lim_{t \to 0} \mathrm{Re}(y, x+ty)_L = \mathrm{Re}(y, x)_L, \quad x, y \in X,$$

a characterization of smooth normed linear spaces will be established ([1] p. 387).

THEOREM B. *Let* $(X, \|\cdot\|)$ *be a normed linear space and let* $(\cdot,\cdot)_L$ *be an L-semi-inner-product which generates the norm* $\|\cdot\|$. *Then* $(\cdot,\cdot)_L$ *is continuous iff* $(X, \|\cdot\|)$ *is a smooth normed linear space.*

DEFINITION 2 ([7], [1] p. 389). Let $(X, \|\cdot\|)$ be a real normed linear space and let $f: X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$. Then the mapping.

$$(x, y)_T = (V_+ f)(y) \cdot x = \lim_{t \downarrow 0} \frac{f(y + tx) - f(y)}{t}$$

$x, y \in X$, is called a semi-inner-product in the sense of Tapia, or a T-semi-inner-product.

Some properties of the T-semi-inner-product are given by ([1] p. 390):

(i) $(x, x)_T = \|x\|^2$, $x \in X$;

(ii) $|(x, y)_T| \leq \|x\| \|y\|$, $x, y \in X$;

(iii) $(\alpha x, \beta y)_T = \alpha\beta (x, y)_T$ if $\alpha\beta \geq 0$ and $x, y \in X$;

(iv) the T-semi-inner-product is subadditive and continuous in the first argument.

THEOREM C ([6], [1] p. 392). *Let* $(X, \|\cdot\|)$ *be a real normed linear space and let* ε *be the set of all L-semi-inner-products which generate the norm* $\|\cdot\|$. *Then we have*

$$(2) \qquad (x, y)_T = \sup_{(\cdot,\cdot)_L \in \varepsilon} (x, y)_L, \quad x, y \in X.$$

COROLLARY ([1] p. 292). *The semi-inner-product in the sense of Tapia is an L-semi-inner-product which generates the norm* $\|\cdot\|$ *iff* $(X, \|\cdot\|)$ *is a smooth normed linear space.*

It is well known that the normed linear space $(X, \|\cdot\|)$ is a smooth space iff the norm $\|\cdot\|$ is a Gâteaux differentiable function on $X - \{0\}$.

THEOREM D. ([1] p. 392). *Let* $(X, \|\cdot\|)$ *be a real normed linear space. The following sentences are equivalent:*

(i) *the norm* $\|\cdot\|$ *is Gâteaux differentiable on* $X - \{0\}$;

(ii) $(x, \alpha y)_T = \alpha (x, y)_T$, $\alpha \in \mathbb{R}$, $x, y \in X$;

(iii) $(\alpha x, y)_T = \alpha (x, y)_T$, $\alpha \in \mathbb{R}$, $x, y \in X$;

(iv) $(\alpha x + \beta y, z)_T = \alpha (x, z)_T + \beta (y, z)_T$, $\alpha, \beta \in \mathbb{R}$, $x, y \in X$.

Further on, we shall present the representation theorem for the continuous linear functional on a real normed space established in [7] by R. A. Tapia ([1] p. 400).

THEOREM E. *Let* $(X, \|.\|)$ *be a real normed linear space. Then the following sentences are equivalent:*

(i) $(X, \|\cdot\|)$ *is a smooth reflexive Banach space;*

(ii) *for every* $f \in X^*$, *there exists an* $x_f \in X$ *such that*

(3) $f(x) = (x, x_f)_T$, $\|f\| = \|x_f\|$, $x \in X$

In ref. [6], I. Roşca introduced the following definition (see also [1], p. 401):

DEFINITION 3. An L-semi-inner product $(\cdot,\cdot)_L$ has the Riesz-representation property iff for every $f \in X^*$ there exists an $x_f \in X$ such that

$$(4) \qquad f(x) = (x, x_f)_L, \ \|f\| = \|x_f\|, \ x \in X.$$

Finally, we present the representation theorem due to I. Roşca [6] (see also [1], p. 401):

THEOREM F. *Let* $(X, \|\cdot\|)$ *be a real normed linear space. Then the following assertions are equivalent:*

(i) *There exists a surjective section* $\tilde{J}$ *of duality mapping on* X;

(ii) *there exists an L-semi-inner-product on* X *which generates the norm and which has the Riesz-representation property.*

COROLLARY. *Let* $(X, \|\cdot\|)$ *be a real normed linear space. Then the following assertions are equivalent:*

(i) $(X, \|\cdot\|)$ *is a smooth reflexive Banach space;*

(ii) *there exists a unique L-semi-inner-product on* X *which generates the norm* $\|\cdot\|$ *and which has the Riesz-representation property.*

1. REPRESENTATION THEOREMS IN SMOOTH REFLEXIVE BANACH SPACES

In this section, $(X, \|\cdot\|)$ will be a normed linear space over $\mathbb{K}$, where $\mathbb{K}$ is the real or complex number field.

1.1. LEMMA. *Let* $(X, \|\cdot\|)$ *be a normed linear space and let* $(\cdot,\cdot)_L$ *an L-semi-inner-product which generates the norm* $\|\cdot\|$. *Then the following sentences are equivalent:*

(i) $(X, \|\cdot\|)$ *is a smooth normed linear space;*

(ii) *for every* $x, y \in X$, *there exist the limits*

$$(1) \qquad \lim_{t \to 0} \operatorname{Re}(y, x + ty)_L, \ \lim_{t \to 0} \frac{\operatorname{Re}(x, x + ty)_L - \gamma x, x)_L}{t}.$$

Proof. "(i) $\Rightarrow$ (ii)". If $(X, \|\cdot\|)$ is a smooth normed linear space, then $\lim_{t \to 0} \operatorname{Re}(y, x + ty)_L = \operatorname{Re}(y, x)_L$, for every $x, y \in X$.

On the other hand, we have

$$(2) \qquad \frac{\|x + ty\|^2 - \|x\|^2}{t} = \frac{\operatorname{Re}(x, x + ty)_L - (x, x)_L}{t} + \operatorname{Re}(y, x + ty)_L$$

for every $x, y \in X$ and $t \in \mathbb{R}$, $t \neq 0$.

Since the norm is Gâteaux-differentiable on $X - \{0\}$, it results that the limit $\lim_{t\to 0} \frac{\operatorname{Re}(x, x+ty)_L - (x, x)_L}{t}$ exists for every $x, y \in X$.

"ii) $\Rightarrow$ (i)". It is evident by relation (2).

1.2. Lemma. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and let $(\cdot,\cdot)_L$ be the L-semi-inner product which generates the norm $\|\cdot\|$. Then we have*

$$(3) \qquad (y, x)_T = \operatorname{Re}(y, x)_L = \lim_{t\to 0} \frac{\operatorname{Re}(x, x+ty)_L - \|x\|^2}{t}$$

for every $x, y \in X$.

Proof. Let us consider the mapping $[\cdot,\cdot]_L : X\times X \to \mathbb{R}$ given by $[y, x]_L = \operatorname{Re}(y, x)_L$. Then $[\cdot,\cdot]_L$ is a real L-semi-inner-product which generates the norm. Since $(X, \|\cdot\|)$ is a smooth normed linear space, it results that $[y, x]_L = (y, x)_T$ for every $x, y \in X$.

On the other hand, by relation (2), we have

$$2(y, x)_T = \lim_{t\to 0} \frac{\operatorname{Re}(x, x+ty)_L - \|x\|^2}{t} + [y, x]_L$$

from where (3) is immediate.

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$ and let $(\cdot,\cdot)_L$ be an L-semi-inner-product which generates the norm $\|\cdot\|$.

1.3. Definition ([6], [1] p. 401). The element $x \in X$ is called L-orthogonal over the element $y \in X$ iff $(y, x)_L = 0$. We note that xLy.

The following orthogonality properties in the sense of Lumer are evident:

(i) $\quad 0Lx, \quad xL0, \quad x \in X$;

(ii) $\quad xLx \Rightarrow x = 0$;

(iii) $\quad \alpha \in \mathbb{K}, \; x\,Ly \Rightarrow \alpha x\, Ly$.

1.4. Lemma. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and let $(\cdot,\cdot)$ be the L-semi-inner-product which generates the norm $\|\cdot\|$. If for every $\lambda \in \mathbb{K}$ we have*

$$(4) \qquad \|x + \lambda y\| \geq \|x\|,$$

then

$$(5) \qquad xLy.$$

Proof. a. Let $\lambda = t \in \mathbb{R}$. Then, we have

$$\|x + ty\|^2 - \|x\|^2 \geq 0, \quad t \in \mathbb{R}.$$

Applying the L-semi-inner product properties, one gets

$$(6) \qquad \operatorname{Re}(x, x+ty)_L - (x, x)_L + t\operatorname{Re}(y, x+ty)_L \geq 0, \quad t \in \mathbb{R}.$$

If $t > 0$, we have

$$\frac{\operatorname{Re}(x, x+ty)_L - (x, x)_L}{t} + \operatorname{Re}(y, x+ty)_L \geq 0.$$

If $t \to 0$, $t > 0$, we obtain $\operatorname{Re}(y, x)_L \geq 0$.

Let $t = -\mu$, $\mu > 0$ in (6). Then we obtain

$$\operatorname{Re}(x, x+\mu(-y))_L - (x, x)_L + \mu \operatorname{Re}(-y, x) + \mu(-y))_L \geq 0$$

from where that $\operatorname{Re}(-y, x)_L \geq 0$ is immediate.

But $\operatorname{Re}(-y, x)_L = -\operatorname{Re}(y, x)_L$ which implies that $\operatorname{Re}(y, x)_L = 0$.

b. Let $\lambda = it$, $t \in \mathbb{R}$. Then we have

$$(7) \qquad \operatorname{Re}(x, x+t(iy))_L - (x, x)_L + t\operatorname{Re}(iy, x+t(iy))_L \geq 0, \quad t \in \mathbb{R}.$$

Putting $z := iy$, from (7) we obtain

$$\operatorname{Re}(x, x+tz)_L - (x, x)_L + t\operatorname{Re}(z, x+tz)_L \geq 0, \quad t \in \mathbb{R},$$

which implies that $\operatorname{Re}(z, x)_L = 0$.

But $\operatorname{Re}(iy, x)_L = -\operatorname{Im}(y, x)_L$, and then $\operatorname{Im}(y, x)_L = 0$. We obtain $(y, x)_L = 0$ i.e. xLy. The lemma is proven.

1.5. Definition. Let $(X, \|\cdot\|)$ be a normed linear space and let $(\cdot,\cdot)_L$ be an L-semi-inner-product which generates the norm $\|\cdot\|$. If $A \subset X$ is a non-empty set, then we denote by A^L the set given by

$$A^L := \{x : x \in X \text{ and } xLy, \text{ for every } y \in A\}$$

It is easy to see that $0 \in A^L$; $A \cap A^L = 0$ if $0 \in A$ and $\alpha \in \mathbb{K}$, $x \in A^L$ implies that $\alpha x \in A^L$. We remark, that in general, A^L is not a linear subspace in X.

Further on, we shall establish a representation theorem for the elements of a smooth reflexive Banach space which generalize a well-known result at work in Hilbert spaces.

1.6. Theorem. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space and let $(\cdot,\cdot)_L$ be the L-semi-inner-product which generates the norm $\|\cdot\|$. Then for every E a closed linear subspace in X, and all $x \in X$, there exist $x' \in E$ and $x'' \in E^L$ such that*

$$(8) \qquad x = x' + x''.$$

Proof. Let E be a closed linear subspace in X and let x be an element of X.

If $x \in E$, then $x = x' + x''$ with $x' = x \in E$ and $x'' = 0 \in E^L$.

If $x \notin E$, then there exists an element $x' \in E$ such that $d(x, E) = d(x, x') = \|x - x'\|$. Putting $x'' = x - x'$, we obtain

$$\|x'' + \lambda y\| = \|x - x' + \lambda y\| = \|x - (x' - \lambda y)\| \geq \|x - x'\| = \|x''\|$$

for every $y \in E$ and $\lambda \in \mathbb{K}$.

Applying lemma 1.4.L. we obtain $x''Ly$, for every $y \in E$, which means that $x'' \in E^L$.

1.7. REMARK. If E is a proper closed linear subspace in X, then there exists an $x_0 \in E^{\mathrm{L}}$ such that $x_0 \neq 0$.

We can now prove a representation theorem for the continuous linear functionals on a smooth reflexive Banach spaces over the real or complex number field.

1.8. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space and let $(\cdot,\cdot)_{\mathrm{L}}$ be the L-semi-inner-product which generate the norm $\|\cdot\|$. Then, for every $f \in X^*$, there exists a $u_f \in X$ such that*

$$f(x) = (x, u_f)_{\mathrm{L}}, \quad \|f\| = \|u_f\|, \quad x \in X. \tag{9}$$

In addition, if $f \neq 0$, then the representation element u_f is given by

$$u_f = \frac{\overline{f(w)}}{\|w\|^2}\, w \tag{10}$$

where $w \in \mathrm{Ker}(f)^{\mathrm{L}}$ and $w \neq 0$.

Proof. Let $f \in X^*$. If $f = 0$ and putting $u_f = 0$, then relation (9) is satisfied. If $f = 0$, then $\ker(f)$ is a proper closed linear subspace in X, and there exists a $w_0 \in \mathrm{Ker}(f)^{\mathrm{L}}$ such that $w_0 \neq 0$.

Let $w \in \mathrm{Ker}(f)^{\mathrm{L}}$, $w \neq 0$ and $x \in X$. Then, we have

$$f(x)w - f(w)\, x \in \mathrm{Ker}(f),$$

which implies that $w\mathrm{L}(f(x)w - f(w)x)$. We obtain

$$(f(x)w - f(w)\, x, w)_{\mathrm{L}} = 0$$

for every $x \in X$, from where it results that $f(x) = \dfrac{f(w)}{\|w\|^2}\,(x, w)_{\mathrm{L}}$, $x \in X$.

Putting $u_f = \dfrac{\overline{f(w)}}{\|w\|^2}\, w$, it results that $f(x) = (x, u_f)_{\mathrm{L}}$, $x \in X$.

On the other hand, we have

$$|f(x)| = |(x, u_f)_{\mathrm{L}}| \leqslant \|x\|\,\|u_f\| \quad \text{and}$$

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geqslant \frac{|f(u_f)|}{\|u_f\|} = \|u_f\|$$

which implies that $\|f\| = \|u_f\|$.

The theorem is proven.

1.9. OBSERVATION. This theorem extends the implication "(i)⇒ (ii)" of Tapia's theorem to the complex case and provides more information in connection with the representation element u_f. We also note that if X is a Hilbert space, one gets the Riesz theorem with a similar proof.

1.10. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space over a complex field. Then, for every $f \in X^*$, there exists a $u_f \in X$ such that*

$$f(x) = (x, u_f)_{\mathrm{T}} - i(ix, u_f)_{\mathrm{T}}, \quad \|f\| = \|u_f\|, \quad x \in X. \tag{11}$$

Proof. It is easy to see that $(x, y)_{\mathrm{L}} = \mathrm{Re}(x, y)_{\mathrm{L}} - \mathrm{i}\,\mathrm{Re}(ix, y)_{\mathrm{L}}$ for every $x, y \in X$. But $\mathrm{Re}(x, y)_{\mathrm{L}} = (x, y)_{\mathrm{T}}$ (see lemma 1.2.L), and applying theorem 1.8.T, the corollary is proven.

1.11. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space over a complex field and let $(\cdot,\cdot)_{\mathrm{L}}$ be the L-semi-inner-product which generates the norm $\|\cdot\|$. Then, for every $f \in X^*$, there exists a $u_f \in X$ such that*

$$f(x) = \lim_{t \to 0} \frac{\mathrm{Re}(u_f, u_f + tx)_{\mathrm{L}} - \|u_f\|^2}{t} - \mathrm{i} \lim \frac{\mathrm{Re}(u_f, u_f + \mathrm{i}t\, x)_{\mathrm{L}} - \|u_f\|^2}{t} \tag{12}$$

$x \in X$, and

$$\|f\| = \|u_f\|. \tag{13}$$

Proof. By lemma 1.2.L, we have

$$(x, y)_{\mathrm{T}} = \lim_{t \to 0} \frac{\mathrm{Re}(y, y + tx)_{\mathrm{L}} - \|y\|^2}{t}$$

and applying corollary 1.10, we obtain (12).

2. REPRESENTATION THEOREMS IN SMOOTH REFLEXIVE BANACH SPACES OF (N)-TYPE

Let $(X, \|\cdot\|)$be a smooth normed linear space over $\mathbb{K}(\mathbb{K} = \mathbb{R}, \mathbb{C})$.

2.1. DEFINITION. The space $(X, \|\cdot\|)$ is called a smooth normed linear space of (N)-type of the L-semi-inner-product which generates the norm satisfy the relation

$$|(x, y + z)_{\mathrm{L}}| \leqslant |(x, y)_{\mathrm{L}}| + |(x, z)_{\mathrm{L}}| \tag{N}$$

for every $x, y, z \in X$.

Since $(X, \|\cdot\|)$ is a smooth normed linear space, we note that condition (N) is equivalent with the condition

$$|\langle J(y + z), x\rangle| = |\langle J(y), x\rangle| + |\langle J(z), x\rangle|, \quad x, y, z \in X \tag{J}$$

where J is the duality mapping on X.

2.2. REMARK. Every prehilbertian space is a smooth normed linear space of (N)-type

2.3. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space of (N)-type. Then, for every E closed linear subspace in X, we have :*

(i) *E^{L} is a closed linear subspace in X,*

(ii) *for every $x \in X$, there exists a unique $x' \in E$ and a unique $x'' \in E^{\mathrm{L}}$ such that $x = x' + x''$.*

(iii) *$X = E \oplus E^{\mathrm{L}}$ (a topological sum).*

Proof. (i) Suppose that $x, y \in E^L$. Then, for every $e \in E$, we have $|(e, x+y)_L| \leqslant |(e, x)_L| + |(e, y)_L| = 0$, which implies that $x + y \in E^L$.

Since $\alpha \in \mathbb{K}$, $x \in E^L \Rightarrow \alpha x \in E^L$, one gets that E^L is a linear subspace in X.

Let us consider the mapping $p_e : X \to \mathbb{R}$, $p_e(x) = |(e, x)_L|$, where $e \in X$ and $e \neq 0$. It is easy to see that p_e is a semi-norm on X, for every $e \in X$, $e \neq 0$.

Let $x_n \to x$ in X. Then

$$||(e, x_n)_L| - |(e, x)_L|| = |p_e(x_n) - p_e(x)| \leqslant p_e(x_n - x) =$$

$$= |(e, x_n - x)_L| \leqslant \|e\| \, \|x_n - x\|, \text{ what implies } |(e, x_n)_L| \to |(e, x)_L|.$$

If $y_n \in E^L$ and $y_n \to y$ in X, then, for every $e \in E$, we have

$$0 = |(e, y_n)_L| = \lim_{n\to\infty} (|e, y_n)_L|) = |(e, \lim_{n\to\infty} y_n)_L| = |(e, y)_L|,$$

which means that $y \in E^L$.

Consequently, E^L is a closed linear subspace in X.

(ii). Let $x \in X$ and let

$$x = x' + x'', \; x' \in E, \; x'' \in E^L;$$

$$x = y' + y'', \; y' \in E; \; y'' \in E^L$$

be two representations of x.

One gets

$$E \ni x' - y' = x'' - y'' \in E^L.$$

Since $E \cap E^L = \{0\}$, we obtain $x' = y'$, $x'' = y''$.

(iii). It is evident by (i) and (ii).

The theorem is proven.

Another property of smooth reflexive Banach spaces which satisfies condition (N) is included in the following theorem.

2.4. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space of (N)-type and let $(\cdot,\cdot)$ be the L-semi-inner-product which generates the norm $\|\cdot\|$. Then, for every $f \in X^*$, there exists a unique element $u_f \in X$ such that*

$$(1) \qquad f(x) = (x, u_f)_L, \quad \|f\| = \|u_f\|, \quad x \in X.$$

In addition, if $f \neq 0$, u_f is given by

$$(2) \qquad u_f = \frac{\overline{f(w)}}{\|w\|^2} w,$$

where $w \in \mathrm{Ker}(f)^L$ and $w \neq 0$.

Proof. Let $f \in X^*$.

If $f = 0$ and if we suppose that $0 = f(x) = (x, v_f)_L$ with $x \in X$ and $v_f \neq 0$, we obtain $0 = f(v_f) = \|v_f\|^2$ which implies that $v_f = 0$.

If $f \neq 0$, then by theorem 1.8.T, there exists an element $u_f \neq 0$ such that relation (1) is valid. We have $(x, u_f)_L = f(x) = 0$ for every $x \in \mathrm{Ker}(f)$ which implies that $u_f \in \mathrm{Ker}(f)^L$.

If $v_f \in \mathrm{Ker}(f)^L$ is another element which satisfies relation (1), since $\mathrm{Ker}(f)^L$ is a one-dimensional linear subspace in X, there exists a $\lambda \in \mathbb{K}$, $\lambda \neq 1$, such that $v_f = \lambda u_f$.

One gets

$$(x, u_f)_L = f(x) = (x, v_f)_L = \overline{\lambda}(x, u_f)_L, \quad x \in X$$

from where it results that $\lambda = 1$.

The theorem is proven.

2.5. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space of (N)-type over the complex number field. Then, for every $f \in X^*$, there exists a unique element $u_f \in X$ such that*

$$(3) \qquad f(x) = (x, u_f)_T - \mathrm{i}(\mathrm{i}x, u_f)_T, \quad \|f\| = \|u_f\|, \quad x \in X.$$

2.6. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space of (N)-type over the complex number field and let $(\cdot,\cdot)_L$ be the L-semi-inner-product which generates the norm $\|\cdot\|$. Then, for every $f \in X^*$, there exists a unique element $u_f \in X$ such that*

$$(4) \qquad f(x) = \lim_{t\to 0} \frac{\mathrm{Re}(u_f, u_f + tx)_L - \|u_f\|^2}{t} - \mathrm{i} \lim_{t\to 0} \frac{\mathrm{Re}(u_f, u_f + \mathrm{i}tx)_L - \|u_f\|^2}{t}$$

for every $x \in X$, and

$$(5) \qquad \|f\| = \|u_f\|.$$

3. APPLICATIONS

In ref. [2] M. Golomb and R. A. Tapia (see also [1], p. 283) proved the following theorem:

3.1. THEOREM. *Let $(X, \|\cdot\|)$ be a real Banach space with a univocal duality mapping. Then J is a linear operator if and only if $(X, \|\cdot\|)$ is a prehilbertian space.*

Further on, we shall give a characterization theorem for the Hilbert space by use of lemma 1.2.L.

3.2. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth Banach space. Then the following sentences are equivalent:*

(i) *$(X, \|\cdot\|)$ is a Hilbert space;*

(ii) *for every $x, y \in X$, we have*

$$\lim_{t\to 0} \frac{\mathrm{Re}(x, x + ty)_L - \|x\|^2}{t} = \mathrm{Re}(x, y)_L,$$

where $(\cdot,\cdot)_L$ is the L-semi-inner-product which generates the norm.

Proof. "(i) → (ii)". It is evident.

"(ii) ⇒ (*i*)". By lemma 1.2.L, we have

$$\mathrm{Re}(y, x)_L = \lim_{t \to 0} \frac{\mathrm{Re}(x, x + ty)_L - \|x\|^2}{t}, \quad x, y \in X$$

which implies that $\mathrm{Re}\,(y, x)_L = \mathrm{Re}(x, y)_L$.

If $(X, \|\cdot\|)$ is a smooth Banach space over the real number field, then the theorem is proven.

If $(X, \|\cdot\|)$ is a smooth Banach space over the complex number field, we have

$$\begin{aligned}(y, x)_L &= \mathrm{Re}(y, x)_L + \mathrm{i}\,\mathrm{Im}(y, x)_L = \mathrm{Re}(y, x)_L + \mathrm{i}\,\mathrm{Re}[-\mathrm{i}(y, x)_L] = \\ &= \mathrm{Re}(y, x)_L - \mathrm{i}\,\mathrm{Re}(\mathrm{i}y, x)_L = \mathrm{Re}(y, x)_L - \mathrm{i}\,\mathrm{Re}(x, \mathrm{i}y)_L = \\ &= \mathrm{Re}(x, y)_L - \mathrm{i}\,\mathrm{Re}[\mathrm{i}(x, y)_L] = \mathrm{Re}(x, y)_L + \mathrm{i}\,\mathrm{Re}[\mathrm{i}(x, y)_L] = \\ &= \mathrm{Re}(x, y)_L - \mathrm{i}\,\mathrm{Im}(x, y)_L = \overline{(x, y)}_L\end{aligned}$$

for every $x, y \in X$, which implies that $(\cdot, \cdot)_L$ is a scalar product on X.

The theorem is proven.

The theorem of Lindenstrauss-Tzafriri [3] (see also [5], p. 198) is well known:

3.3. THEOREM. *Let $(X, \|\cdot\|)$ be a Banach space. If every $E \subset X$, a closed linear subspaces, is complemented in X, then $(X, \|\cdot\|)$ is isomorphic to a Hilbert space.*

Finally, using the Lindenstrauss-Tzafriri theorem, we can prove the following theorem:

3.4. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space of (N)-type. Then $(X, \|\cdot\|)$ is isomorphic to a Hilbert space.*

The proof results from theorem 2.3.T.

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