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AN INEQUALITY WHICH IS NOT AN ABSTRACT GEOMETRIC INEQUALITY

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The theory of extended geometric programming is based on the abstract geometric inequalities introduced by R. J. Duffin, E. L. Peterson and C. Zener in ref. [4].

DEFINITION 1. An inequality is said to be an abstract geometric inequality if it satisfies the following postulates:

(*i*) The inequality is a scalar product inequality of the form

$$\sum_{i=1}^{n} x_i y_i \leq \lambda(y)\, G(x) - F(y), \tag{1}$$

which is valid for each vector $x = (x_1, \ldots, x_n)$ in an open convex set $C \subseteq R^n$ and each vector $y = (y_1, \ldots, y_n)$ in a cone $K \subseteq R^n$, where $F, \lambda : K \to R$ and $G : C \to R$ are functions.

(*ii*) For any vector $x \in C$, there is a nonzero vector $z \in K$ such that inequality (1) becomes an equality for each vector y on the ray emanating from the origin through the point z, i.e.

$$\sum_{i=1}^{n} x_i y_i = \lambda(y)\, G(x) - F(y), \text{ for all } y = \alpha z,\ \alpha \geqslant 0.$$

(*iii*) The function λ is nonnegative on the set K.

(*iv*) The function G is differentiable on the open convex set C.

The abstract geometric inequalities have some useful properties stated in the following lemmas [4].

LEMMA 1. *If an abstract geometric inequality is actually an equality for a vector x in C and a vector y in K, then*

$$y_i = \lambda(y) \frac{\partial G}{\partial x_i}(x), \textit{ for any } i \in \{1, \ldots, n\}.$$

LEMMA 2. *If x is a vector in C and α is an arbitrary non-negative number, then the vector*

$$y = \alpha \nabla G(x)$$

is in K and the abstract geometric inequality (1) *becomes an equality for x and y. Moreover, $y \neq 0$ when $\alpha \neq 0$.*

LEMMA 3. *For each abstract geometric inequality and for each non-negative number α, the following identities are valid:*

$$\lambda(\alpha \nabla G(x)) = \alpha, \quad \text{for all} \quad x \in C \tag{2}$$

and

$$F(\alpha \nabla G(x)) = \alpha F(\nabla G(x)), \quad \text{for all} \quad x \in C. \tag{3}$$

LEMMA 4. *The function G appearing in an abstract geometric inequality is a convex function.*

Duffin, Peterson and Zener [4] showed that the inequality relating the arithmetic and geometric means, the inequality relating the arithmetic and harmonic means, Hölder's inequality and several other inequalities are special cases of abstract geometric inequalities. Through these inequalities, generalizations of geometric programming are obtained.

In [2], Duca established an inequality given by the following lemma.

LEMMA 5. *Let $u_1 \geq 0, \ldots, u_n \geq 0$ and $y_1 \geq 0, \ldots, y_n \geq 0$, $\sum_{i=1}^{n} y_i \neq 0$. Then*

$$\sum_{i=1}^{n} u_i \geq \ln \left[\left(\sum_{i=1}^{n} y_i \right) \prod_{i=1}^{n} \left(\frac{e u_i}{y_i} \right)^{y_i \Big/ \sum_{i=1}^{n} y_i} \right], \tag{4}$$

the equality being valid if and only if

$$u_i = y_i \Big/ \left(\sum_{i=1}^{n} y_i \right), \quad \text{for all } i \in \{1, \ldots, n\}.$$

The p r o o f is given in ref. [2].

Here and for what follows, we define $t \ln t = 0$ if $t = 0$.

In [2], we have developed a duality theory of geometric programming based on inequality (4).

In this paper, we shall show that inequality (4) can be written as a scalar product inequality of the form (1) and we shall prove that the inequality obtained is not an abstract geometric inequality. This proves that the duality theory developed in ref. [2] is not an individual case of the duality theory developed in ref. [4].

THEOREM 1. (*i*) *Let $x = (x_1, \ldots, x_n)$ in R^n and let*

$$y = (y_1, \ldots, y_n) \geq 0 \text{ with } \sum_{i=1}^{n} y_i \neq 0. \text{ Then}$$

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} y_i \right) \left(\sum_{i=1}^{n} e^{x_i} \right) + \sum_{i=1}^{n} y_i \ln y_i - \left(\sum_{i=1}^{n} y_i \right) \ln \left(\sum_{i=1}^{n} y_i \right) - \sum_{i=1}^{n} y_i, \tag{5}$$

the equality being valid if and only if

$$e^{x_j} \left(\sum_{i=1}^{n} y_i \right) = y_j, \text{ for all } j \in \{1, \ldots, n\}.$$

(*ii*) *Inequality* (5) *is not an abstract geometric inequality.*

Proof. (*i*) Part (*i*) of the theorem is obtained applying lemma 5 to the system of numbers:

$$u_1 = e^{x_1}, \ldots, u_n = e^{x_n}; \ y_1 \geq 0, \ldots, y_n \geq 0 \text{ with } \sum_{i=1}^{n} y_i \neq 0.$$

(*ii*) Inequality (5) can be expressed as a scalar product inequality of the form (1), if in definition 1 we take $C = R^n$, the cone $K = R^n_+$ as the non-negative orthant of R^n and the functions $F, \lambda : K \to R$, $G : C \to R$ defined by

$$F(y) = \left(\sum_{i=1}^{n} y_i \right) \ln \left(\sum_{i=1}^{n} y_i \right) - \sum_{i=1}^{n} y_i \ln y_i + \sum_{i=1}^{n} y_i, \quad y \in K,$$

$$\lambda(y) = \sum_{i=1}^{n} y_i, \quad y \in K,$$

$$G(x) = \sum_{i=1}^{n} e^{x_i}, \quad x \in C.$$

Evidently, in this case, the postulates (*i*), (*iii*) and (*iv*) of definition 1 are fulfilled. We shall show that the postulate (*ii*) of definition 1 is not fulfilled. We shall show this by contradiction. Assume that postulate (*ii*) is fulfilled. Let

$x = (x_1, \ldots, x_n) \in C$ with $\sum_{i=1}^{n} e^{x_i} \neq 1$. Then, by postulate (*ii*), for the above x, there exists a nonzero vector $z \in K$ so that

$$\sum_{i=1}^{n} x_i y_i = \lambda(y)\, G(x) - F(y) \text{ for all } y = \alpha z, \ \alpha \geqslant 0.$$

By statement (*i*) of theorem 1, the equality in (5) holds if and only if

$$e^{x_j} \sum_{i=1}^{n} \alpha z_i = \alpha z_j \text{ for all } j \in \{1, \ldots, n\}.$$

Assume that α is strictly positive. Then the inequality in (5) is valid if and only if

$$e^{x_j} \sum_{i=1}^{n} z_i = z_j \text{ for all } j \in \{1, \ldots, n\},$$

i.e., if and only if the coordonates of the vector $z = (z_1, \ldots, z_n)$ are a solution to the linear and homogeneous system

$$
(6)\qquad \begin{cases}
(e^{x_1}-1)\,z_1 + e^{x_1}\quad z_2 + \dots + e^{x_1}z_n = 0\\
e^{x_2}\quad z_1 + (e^{x_2}-1)z_2 + \dots + e^{x_2}z_n = 0\\
\dots\dots\dots\dots\dots\dots\dots\dots\\
e^{x_n}\quad z_1 + e^{x_n}\quad z_2 + (e^{x_n}-1)\,z_n = 0.
\end{cases}
$$

The determinant of system (6) is

$$
D = \begin{vmatrix}
e^{x_1}-1 & e^{x_1} & \dots & e^{x_1} & e^{x_1}\\
e^{x_2} & e^{x_2}-1 & \dots & e^{x_2} & e^{x_2}\\
\dots & \dots & \dots & \dots & \dots\\
e^{x_{n-1}} & e^{x_{n-1}} & \dots & e^{x_{n-1}}-1 & e^{x_{n-1}}\\
e^{x_n} & e^{x_n} & \dots & e^{x_n} & e^{x_n}-1
\end{vmatrix} =
$$

$$
= \begin{vmatrix}
-1 & 0 & 0 & \dots & 0 & e^{x_1}\\
0 & -1 & 0 & \dots & 0 & e^{x_2}\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & & -1 & e^{x_{n-1}}\\
1 & 1 & 1 & \dots & 1 & e^{x_n}-1
\end{vmatrix} =
\begin{vmatrix}
-1 & 0 & 0 & \dots & 0 & e^{x_1}\\
0 & -1 & 0 & \dots & 0 & e^{x_2}\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & \dots & -1 & e^{x_{n-1}}\\
0 & 0 & 0 & \dots & 0 & \sum_{i=1}^{n} e^{x_i}-1
\end{vmatrix} =
$$

$$
= (-1)^{n-1}\left(\sum_{i=1}^{n} e^{x_i} - 1\right).
$$

Since $\sum_{i=1}^{n} e^{x_i} \neq 1$, we have $D \neq 0$; thus, system (6) admits only the trivial solution $z_1 = z_2 = \dots = z_n = 0$. Then the vector $z = 0$, which contradicts the hypothesis that $z \neq 0$. Thus, axiom (*ii*) of definition 1 is not verified, so that inequality (5) is not an abstract geometric inequality and the theorem is proven.

The properties of the abstract geometric inequalities expressed by lemmas 1 and 4 are valid for inequality (5) too. To the properties of the abstract geometric inequalities expressed by lemmas 2 and 3 correspond the properties of inequality (5) given by the following lemmas.

LEMMA 6. *If x is an arbitrary vector in C and α is an arbitrary non-negative number, then the vector $y = \alpha\nabla G(x)$ is in K. Moreover, $y \neq 0$ when $\alpha \neq 0$.*

If the vector x in C has the property that $\sum_{i=1}^{n} e^{x_i} = 1$, then inequality (5) becomes an equality for x and $y = \alpha\nabla G(x)$.

LEMMA 7. *For any non-negative number α, the functions F, λ, and G verify the following identities:*

$$
(7)\qquad \lambda(\alpha\nabla G(x)) = \alpha G(x), \ \textit{for all}\ \ x \in C,
$$

and

$$
(8)\qquad F(\alpha\nabla G(x)) = \alpha F(\nabla G(x)), \ \textit{for all}\ \ x \in C.
$$

The proof of the lemmas is easily obtained from theorem 1 and the definition of the functions F, λ, and G.

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