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COMPUTATIONAL ASPECTS OF SOME ITERATIVE METHODS FOR BOUNDING THE INVERSE OF A MATRIX

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Abstract: We consider iterative methods for improving bounds for the inverse of a matrix. A modification of the well-known higher-order interval Schulz methods is proposed. Its computational efficiency is as for the original methods but it is preferable with regard to the accumulated rounding errors. Furthermore, for practical applications, a sufficient criterion for the monotonicity of the methods is given and thus taking intersections after each step can be avoided.

Let A be an (n, n) real nonsingular matrix and let $X^{(0)}$ be an (n, n) interval matrix with

$$A^{-1} \in X^{(0)}.$$

In ref. [3], Chapter 18, we can find iterative methods for improving $X^{(0)}$ by means of the interval arithmetics. These methods can be considered as interval versions of the higher-order Schulz methods and are defined as follows:

$$(1)\qquad X^{(k+1)} = m(X^{(k)}) \sum_{i=0}^{r-2} (I - Am(X^{(k)}))^i + X^{(k)}(I - Am(X^{(k)}))^{r-1},$$

$$(2)\quad X^{(k+1)} = \left\{ m(X^{(k)}) \sum_{i=0}^{r-2} (I - Am(X^{(k)}))^i + X^{(k)}(I - Am(X^{(k)}))^{r-1} \right\} \cap X^{(k)}.$$

where I denotes the unit-matrix and

$$m(X) = m((X_{ij})) = ((x_{ij}^1 + x_{ij}^2)/2)$$

is the midpoint matrix of an interval matrix X. The integer parameter r is to be greater than 1. It can be shown that

$$A^{-1} \in X^{(k)}, \ k \geqslant 0$$

holds true for (1) and (2). Obviously, for (2) we have the property

$$(3)\qquad X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \ldots,$$

which means that the iterates of (2) are forming a nested sequence of interval matrices. As to the convergence of the methods to A^{-1} it can be shown (compare Theorem 1 and 2 in ref. [3], Chapter 18) that for method (1)

$$\rho(I - Am(X^{(0)})) < 1 \Leftrightarrow \lim_{k\to\infty} X^{(k)} = A^{-1}$$

is valid whereas, for method (2), the stronger condition

$$\rho(|I - AX|) < 1 \text{ for } X \in X^{(0)} \Rightarrow \lim_{k\to\infty} X^{(k)} = A^{-1}$$

holds. The R-order of convergence is measured as usual by the R-order of the sequence $\{\|d(X^{(k)})\|\}$, where

$$d(X) = d((X_{ij})) = (x_{ij}^2 - x_{ij}^1)$$

is the width-matrix of an interval matrix and $\|.\|$ denotes an arbitrary matrix norm. In the above-mentioned Theorems, it is shown that for method (1) as well as for method (2)

$$O_R(A^{-1}, \{X^{(k)}\}) \geqslant r$$

can be estimated.

For practical computations, it seems reasonable to start the iteration — once r has been chosen — with method (1) because the convergence criterion is considerably weaker than that of (2). After some iterations when the condition for the convergence of (2) to A^{-1} is fullfilled, then (2) should be applied because of its monotonicity which leads to a quite natural stopping rule. With regard to the parameter r, it can be shown that $r = 3$ is an optimal choice with respect to the computational efficiency (see ref. [10], Appendix C) when evaluating the formulas in (1) and (2) according to the Horner-scheme. This optimality is, however, not true if we take into account the accumulated rounding errors. To minimize these, the choice $r = 2$ should be taken.

Before dealing with the problem of an optimal method we consider first the aspect of a combined method of (1) and (2). We shall show that the proposed switching from (1) to (2) can be avoided in many cases. This could save a remarkable amount of operations in each step of (1) necessary for the test of the convergence condition for (2) to A^{-1}.

Numerical examples showed that for almost every choice of $X^{(0)}$ in the procedures of [3], Appendix C, the iteration according to (1) has already monotonic behaviour. On the other hand, from the necessary condition for the monotonicity of (1)

(4) $$X^{(0)} \supseteq X^{(1)}$$

it follows that in the case of monotonicity, $d(X^{(0)})$ must be sufficiently large (see [6]). The question remains how large $d(X^{(0)})$ should be chosen so that (1) may be already monotonic. Remember that $d(X^{(0)})$ has no influence on the convergence criterion for (1), which is only a condition on the quality of $m(X^{(0)})$ as an approximation of A^{-1}. An important step in this direction was made by Schmidt in ref. [9], where it is shown that

(4) is also a sufficient condition for the monotonicity of (1). Now we can conclude in the following way. (4) is equivalent with

$$|m(X^{(1)}) - m(X^{(0)})| \leqslant \frac{1}{2}(d(X^{(0)}) - d(X^{(1)})).$$

For the sequence $\{m(X^{(k)})\}$, after a simple rearrangement of terms, we get

$$(I - Am(X^{(1)})) = (I - Am(X^{(0)}))^r.$$

This leads to the equation

$$m(X^{(1)}) = A^{-1}(I - (I - Am(X^{(0)}))^r)$$

and once this is substituted into the above inequality, we have

$$|A^{-1}(I - (I - Am(X^{(0)}))^r) - m(X^{(0)})| \leqslant \frac{1}{2}(d(X^{(0)}) - d(X^{(1)})).$$

Since

$$d(X^{(1)}) = d(X^{(0)})|(I - Am(X^{(0)}))^{r-1}|$$

is valid, we finally get

(5) $$|A^{-1}(I-(I-Am(X^{(0)}))^r)-m(X^{(0)})| \leqslant \frac{1}{2} d(X^{(0)})(I - |(I-Am(X^{(0)}))^{r-1}|)$$

as a necessary and sufficient condition for the monotonicity of (1). This inequality enables us to proof the following

LEMMA. *Let* $A^{-1} \in X^{(0)}$ and let $\|I - Am(X^{(0)})\| < 1$ ($\|.\|$ *denotes the column-sum norm*)*; then* (1) *converges to* A^{-1} *and if* $d(X^{(0)})$ *is chosen according to the rule*

$$d(X_{ij}^{(0)}) = h \geqslant 2 \cdot \frac{(1 + \|I - Am(X^{(0)})\|^r) \cdot \max_{i,j} |b_{ij}| + \max_{i,j} |m(X_{i,j}^{(0)})|}{1 - \|I - Am(X^{(0)})\|^{r-1}}$$

where $B = A^{-1}$*, then* (1) *is monotonic.*

Proof: The proof can be done by simply substituting $d(X_{ij}^{(0)}) = h$ into the right-hand side of (5) and then, estimating the result as below which gives the left-hand side expression of (5).

Remarks: In the above formula for h one can simplify the expression by realizing that

$$\|I - Am(X^{(0)})\|^k < \|I - Am(X^{(0)})\|$$

holds true. For the case when $r = 2$, it is possible to get some sharper bounds for h (see [5]) and it can be shown that in this case the automatically generated matrices $X^{(0)}$ in the procedures given in ref. [2] and [3] always produce monotonic sequences of iterates.

Now, we consider the second problem of constructing an optimal method with respect to the computational efficiency and to the accumulated rounding errors. As was stated earlier, the optimal choices are $r = 3$

and $r = 2$, respectively. The proposed methods herein try to combine both choices by defining higher-order methods which use only formulas comparable with the case when $r = 2$. Our methods are the following ones:

$$
(6)\quad
\begin{aligned}
y^{(k+1,0)} &= m(X^{(k)}) + X^{(k)}(I - Am(X^{(k)})),\\
y^{(k+1,i)} &= m(X^{(k)}) + y^{(k+1,i-1)}(I - Am)(X^{(k)})),\ 1 \leqslant i,\ i \leqslant s,\\
X^{(k+1)} &= m(X^{(k)}) + y^{(k+1,s)}(I - Am(X^{(k)})),\ (s \text{ fixed})
\end{aligned}
$$

$$
(7)\quad
\begin{aligned}
y^{(k+1,0)} &= \{m(X^{(k)}) + X^{(k)}(I - Am(X^{(k)}))\} \cap X^{(k)},\\
y^{(k+1,i)} &= \{m(X^{(k)}) + y^{(k+1,i-1)}(I - Am(X^{(k)}))\} \cap y^{(k+1,i-1)},\ 1 \leqslant i,\ i \leqslant s\\
x^{(k+1)} &= \{m(X^{(k)}) + y^{(k+1,s)}(I - Am(X^{(k)}))\} \cap y^{(k+1,s)} (s \text{ fixed})
\end{aligned}
$$

Again, (7) is the monotonic version of (6). As for the interval Schulz methods, one uses the equality

$$A^{-1} = X + A^{-1}(I - AX)$$

together with the inclusion property in the interval arithmetics to prove by complete induction the properties

$$A^{-1} \in y^{(k,i)},\ 1 \leqslant i \leqslant s,\ A^{-1} \in X^{(k)},\ k \geqslant 0$$

for methods (6) and (7).

On closer inspection, the interval matrices $y^{(k,i)}$ and $X^{(k)}$ in (6) turn out to be just the intermediate results of the Horner-scheme computation of (1). However, this is not true for (7). So, we can conclude that the sequence $\{X^{(k)}\}$ and — as a simple consequence — the sequences $\{y^{(k,i)}\}$ converge to A^{-1} iff $\rho(I - Am(X^{(0)})) < 1$. Also, by direct proof it can easily be shown that

$$\rho(|I - AX|) < 1 \text{ for } X \in X^{(0)}$$

guarantees the convergence of the sequences $\{y^{(k,i)}\}$ and $\{X^{(k)}\}$ of (7) to A^{-1}. If we apply the width operator d to the equations in (6) and (7) and then estimate the right-hand sides as above like for the interval Schulz methods, we get the following system of inequalities:

$$\mathrm{d}(y^{(k+1,0)}) \leqslant \frac{1}{2}\, d(X^{(k)})\,|A|\,\mathrm{d}(X^{(k)})$$

$$\mathrm{d}(y^{(k+1),i)}) \leqslant \frac{1}{2}\, \mathrm{d}(y^{(k+1,i-1)})\,|A|\,\mathrm{d}(X^{(k)}),\ 1 \leqslant i, i \leqslant s,$$

$$\mathrm{d}(X^{(k+1)}) \leqslant \frac{1}{2}\, \mathrm{d}(y^{(k+1,s)})\,|A|\,\mathrm{d}(X^{(k)}).$$

Now, we apply a monotonic and multiplicative matrix norm to these inequalities and because of the equivalence of all matrix norms, we get

the following system of recurrences for an arbitrary norm:

$$\|\mathrm{d}(y^{(k+1,0)})\| \leqslant \alpha^{(0)}\|\mathrm{d}(X^{(k)})\|^2,$$

$$\|\mathrm{d}(y^{(k+1,i)})\| \leqslant \alpha^{(i)}\|\,\mathrm{d}(y^{(k+1,i-1)})\|.\ \|\,\mathrm{d}(X^{(k)})\|,\ 1 \leqslant i,\ i \leqslant s,$$

$$\|d(X^{(k+1)})\| \leqslant \alpha^{(s+1)}\|\,\mathrm{d}(y^{(k+1,s)})\|.\ \|\,\mathrm{d}(X^{(k)})\|.$$

Such a system was treated in ref. [4] and it was proved that we have

$$\mathrm{O}_R(A^{-1}, \{X^{(k)}\}) \geqslant s + 3,\ \mathrm{O}_R(A^{-1}, \{y^{(k,i)}\}) \geqslant s + 3,\ 0 \leqslant i \leqslant s.$$

This means that all the intermediate results of the Horner-scheme $y^{(k,i)}$ behave like $X^{(k)}$.

If we measure the computational efforts in terms of the necessary matrix multiplications in (6) or (7), then we get for the computational efficiency the lower bounds

$$(s + 3)^{\frac{1}{(s+3)}}$$

which have its maximum $\sqrt[3]{3}$ at $s = 0$. The optimal choices with respect to these lower bounds are both methods of order of at least 3:

$$
(6')\quad
\begin{aligned}
y^{(k+1)} &= m(X^{(k)}) + X^{(k)}(I - Am(X^{(k)})),\\
X^{(k+1)} &= m(X^{(k)}) + y^{(k+1)}(I - Am(X^{(k)})),
\end{aligned}
$$

$$
(7')\quad
\begin{aligned}
y^{(k+1)} &= \{m(X^{(k)}) + X^{(k)}(I - Am(X^{k}))\} \cap X^{(k)},\\
X^{(k+1)} &= \{m(X^{(k)}) + y^{(k+1)}(I - Am(X^{(k)}))\} \cap y^{(k+1)}.
\end{aligned}
$$

These methods can be considered as the interval versions of a method proposed in [1]. The accumulated rounding errors are almost the same as for the case $r = 2$ in (1) or (2) because of the similar structure of the corresponding iteration formulas.

For practical purposes, one can again combine (6′) and (7′) in the same way as (1) and (2). Furthermore, for the monotonicity of (6′), the same is true as for (1) with $r = 3$.

We conclude with a numerical example which gives a comparison between a combination of (1) with (2), resp. of (6′) with (7′). For simplicity, we choose $n = 3$.

$$A = \begin{pmatrix} 1 & -0.1 & 0.1\\ -0.1 & 1 & 0.1\\ 0.1 & 0.1 & 1 \end{pmatrix}$$

$$X^{(0)} = \begin{pmatrix} [1,\ 1.2] & [0.1,\ 0.2] & [-0.2,\ 0.1]\\ [0.1,\ 0.2] & [1,\ 1.2] & [-0.2,\ 0.1]\\ [-0.2,\ 0.1] & [-0.2,\ 0.1] & [1,\ 1.2] \end{pmatrix}$$

X is the numerical fixed point of iteration (1) combined with (2). y is the numerical fixed point of iteration (6′) combined with (7′). After a few iterations, in both cases we get the results:

$$d(y) = \begin{pmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{pmatrix} \cdot 10^{-12}$$

$$d(X) = \begin{pmatrix} 20 & 2 & 2 \\ 2 & 20 & 2 \\ 2 & 2 & 20 \end{pmatrix} \cdot 10^{-12}$$

The computations were performed on a microcomputer Apple //e with a PASCAL SC system (see [8]). This example can be found in ref. [7].

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