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ON THE NUMBER OF LIMIT CYCLES FOR SYSTEMS WITH SEVERAL SINGULAR POINTS

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We consider an autonomous system in the case when it has one or more than one singular point. Such systems have been very widely studied and arise frequently in applications. References [2], [4] — [8] give the conditions for a Liénard system to admit k limit cycles surrounding an odd number of singular points. In this paper, we generalize further down the system studied in refs. [4] — [8] and give existence conditions for k limit cycles surrounding $2n+1$ singular points.

Consider the system

$$(1)\qquad \begin{cases} \dot{x} = h(x)\bar{y} - F(x) \\ \dot{\bar{y}} = -g(x) + \alpha(x)\bar{y}, \end{cases}$$

where $F(x) = \int_0^x f(x)\,\mathrm{d}x$, $g(x)$, $f(x)$, $h(x)$, $\alpha(x)$, are continuous functions which satisfy all the conditions ensuring the solution uniqueness for any Cauchy problem. We consider the conditions for $g(x)$

$$g(\alpha_i) = 0,\ i = -1, 0, 1, \ldots, 2n-1;\ \alpha_{-1} < \alpha_0 < \ldots < \alpha_{2n-1},$$

$$(2)\qquad g(x) > 0,\ \forall x \in (\alpha_{2i-3}, \alpha_{2i-2}),\quad i = \overline{1, n},\ \text{and}\ \forall\ x \in (\alpha_{2n-1}, \infty)$$

$$g(x) < 0,\ \forall x \in (\alpha_{2i-2}, \alpha_{2i-1}),\ i = \overline{1, n},\ \text{and}\ \forall x \in (-\infty, \alpha_{-1}).$$

THEOREM 1. *Let the function $g(x)$ satisfy conditions (2). There exist the functions $\varphi_i(x)$, $i = \overline{1,k}$, $k \geq 2$, $[\alpha(x) - \varphi_i'(x)\,h(x)] \neq 0$, the systems of numbers $x_{-k} < x_{-k+1} < \ldots < \ldots x_{-1} < \alpha_{-1}$; $x_k > x_{k-1} > \ldots > x_1 > \alpha_{2n-1}$, and the functions*

$$F_i(x) = F(x) - h(x)\,\varphi_i(x),\ \ \Theta_i(x) = \frac{F(x)\,\varphi_i'(x)}{\alpha(x) - \varphi_i'(x)\,h(x)} - \varphi_i(x) - \frac{g(x)}{\alpha(x) - \varphi_i'(x)h(x)}.$$

We impose the following supplementary conditions:

$$\text{(i)}\quad \varphi_i(0) = 0,\ [\alpha(x) - \varphi_i'(x)\,h(x)]\cdot(-1)^i < 0,\ i = \overline{1, k},$$
$$x \in [x_{-i}, x_i],$$

$$\text{(ii)}\ F_i(x_i)\,(-1)^{i+1} < \varnothing_i(x)(-1)^{i+1} < F_i(x_i)\,(-1)^{i+1},$$

$$x \in [x_{-i},\, x_i],\ i = \overline{1, k}$$

$$h(x_i)F_i(x)\,(-1)^{i+1} \leqslant F_i(x_i)(-1)^{i+1},\ (-1)^{i+1} F_i(x_{-i}) \leqslant$$

$$\leqslant h(x_{-i})\,F_i(x)(-1)^{i+1}\ x \in [x_{-i},\ x_i],$$

$$\text{(iii)}\ F_i(x_i)(-1)^{i+1} > F_i(x_{-i+1})(-1)^{i+1},\ F_i(x_{-i})\,(-1)^i >$$

$$> F_i(x_{i-1})\,(-1)^i,\ i = \overline{1, k}.$$

Then, system (1) *has* $k - 1$ *limit cycles surrounding* $2n + 1$ *singular points. In every domain* $x_{-i} < x < x_i$, $i = \overline{1, k}$, *there are at most* $i - 1$ *limit cycles, out of which* $[i/2]$ *are stable and* $[(i - 1)/2]$ *are unstable.*

Proof. Let $k = 2$. We substitute in (1)

$$(3)\qquad \bar{y} = y + \varphi_1(x).$$

Then system (1) becomes

$$(4)\qquad \begin{cases} \dot{x} = h(x)\bar{y} - F_1(x) \\ \dot{\bar{y}} = +(\alpha(x) - \varphi_1'(x)\,h(x))\,(\bar{y} - \varnothing_1(x)). \end{cases}$$

We build a rectangle Γ_1 having the legs parallel to the coordinate axes, and the vertices $B_1(x_1, F_1(x_1))$, $B_{-1}(x_{-1}, F_1(x_{-1}))$. Also, the trajectories of system (4), which cross the rectangle legs for increasing t penetrate inside the rectangle. Then, we substitute in (4)

$$(5)\qquad \bar{\bar{y}} = y + \varphi_1(x) - \varphi_2(x)$$

and obtain the system

$$(6)\qquad \begin{cases} \dot{x} = h(x)\,\bar{\bar{y}} - F_2(x) \\ \dot{\bar{\bar{y}}} = (\alpha(x) - \varphi_2'(x)\,h(x))\,(\bar{\bar{y}} - \varnothing_2(x)). \end{cases}$$

Let $\bar{\Gamma}_1$ be the closed curve into which the border of the rectangle Γ_1 passes through the transformation (5). The trajectories of system (6) crossing the curve $\bar{\Gamma}_1$ penetrate inside it. Then we build the rectangle Γ_2 having the legs parallel to the coordinate axes, and the vertices $B_2(x_2, F_2(x_2))$, $B_{-2}(x_{-2}, F_2(x_{-2}))$. The trajectories of system (6) crossing the legs of Γ_2 go out of the rectangle. Then, in the ring domain bounded by Γ_2 and $\bar{\Gamma}_1$ there exists at least one unstable limit cycle; therefore, the theorem is proved for $k = 2$. For $k \geqslant 3$, the proof is analogous.

THEOREM 2. *If the conditions of* THEOREM 1 *hold and if we have the supplementary conditions:*

$$\text{(i)}\qquad f(x) < 0,\ \forall x \in (\beta_{-1}, \beta_1);\ f(\beta_{-1}) = f(\beta_1) = 0,$$

$$\beta_{-1} < \alpha_1,\ \beta_1 > \alpha_{2n-1},$$

$$\text{(ii)}\qquad F(\alpha_i) = 0,\ i = -1, 0, 1, \ldots, 2n - 1;\ F(\beta_{-1}) = F(\beta_1) = 0,$$

$$\text{(iii)}\qquad \alpha(x)(y + \varphi_1(x))^2 - \frac{g(x)\,F(x)}{h(x)} > 0,\ \forall x \in (\beta_{-1}, \beta_1),\ x \neq \alpha_i$$

$$\text{(iv)}\qquad \int_0^{\beta_i} \frac{g(x)}{h(x)}\,\mathrm{d}x > 0,\ i = -1;1;\ \int_0^{t} \frac{g(x)}{h(x)}\,\mathrm{d}x \leqslant$$

$$\leqslant \min_{i=-1,1} \int_0^{\beta_i} \frac{g(x)}{h(x)}\,\mathrm{d}x,\ \ \forall t \in (\alpha_1, \alpha_{2n-1})$$

then system* (1) *has at least k limit cycles surrounging $2n + 1$ *singular points,* ***since in every domain*** $x_{-i} < x < x_i$ *there are at least* i *limit cycles, out of which* $[(i + 1)/2]$ *are stable and* $[i/2]$ *are unstable.*

Proof. **The proof is analogous to that in the case of THEOREM 1, the difference consisting of the fact that, here, for the first border we consider the curve:**

$$\frac{(y + \varphi_1(x))^2}{2} + \int_0^x \frac{g(t)}{h(t)} \cdot \mathrm{d}t = C_0$$

where $C_0 = \min_{i=-1,1} \int_0^{\beta_i} \frac{g(t)}{h(t)} \cdot \mathrm{d}t.$

THEOREM 3. *Let the function* $g(x)$ *satisfy conditions* (2). *There exist a function* $\varphi_1(x)$ *and the numbers* $x_{-1} < \alpha_{-1}$, $x_1 > \alpha_{2n-1}$, *such that:*

$$\text{(i)}\qquad \varphi_1(0) = 0,\ [\alpha(x) - \varphi_1'(x)\,h(x)] > 0$$

$$\text{(ii)}\qquad F_1(x_{-1}) < \varnothing_1(x) < F_1(x_1),\ \ \forall x \in (x_{-1}, x_1)$$

$$h(x_1)\,F_1(x) \leqslant F_1(x_1),\ \ F_1(x_{-1}) \leqslant h(x_{-1})\,F_1(x),\ \ x > x_1.$$

Then, all the limit cycles of system (1), *if they exist, are lying in the domain* $x_{-1} < x < x_1$.

Proof. Using a reasoning analogous to the case of THEOREM 1 and ref. [4], the limit cycles of system (1), if they exist, are inside Γ_1.

Remark. If conditions (i) – (iii) in THEOREM 1 are suitably modified, then the limit cycle in the case $k = 2$ is stable.

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