MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 16, N° 1, 1987, pp. 51—53

DUALITY THEOREMS FOR RATIONAL PROGRAMMING PROBLEMS

LIANA LUPŞA
(Cluj-Napoca)

Let $a_{ij}(i=1,\ldots,m,\ j=1,\ldots,n)$, $b_i(i=1,\ldots,m)$ and $c_j(j=1,\ldots,n)$ be rational numbers. For future reference, we define the following sets :

$$X=\{x=(x_1,\ldots,x_n)\in R^n_+ \mid \sum_{j=1}^n a_{ij}x_j \leqslant b_i\ i=1,\ldots,m\},$$

$$Y=\{y=(y_1,\ldots,y_m)\in R^m_+ \mid \sum_{i=1}^m a_{ij}y_i \geqslant c_j\, j=1,\ldots,n\},$$

$$XQ=X\cap Q^n,\ YQ=Y\cap Q^m,$$

where Q is the set of rational numbers.

Let $f: R^n\to R$ and $g: R^m\to R$ be defined by

$$f(x)=\sum_{j=1}^n c_jx_j \text{ for all } x=(x_1,\ldots,x_n)\in R^n,$$

$$g(y)=\sum_{i=1}^m b_iy_i \text{ for all } y=(y_1,\ldots,y_m)\in R^m.$$

We denote by (PQ) the problem

$$(\mathrm{PQ})\begin{cases} f(x)\to \max\\ x\in XQ\end{cases}$$

and by (DQ) the problem

$$(\mathrm{DQ})\begin{cases} g(y)\to \min\\ y\in YQ.\end{cases}$$

In this paper, the duality properties of these problems (PQ) and (DQ) are studied.

REMARK 1. The sets XQ and YQ are polytopes and, because a_{ij} $(i=1,\ldots,n,\ j=1,\ldots,n)$, $b_i(i=1,\ldots,m)$ and $c_j(j=1,\ldots,n)$ are rational numbers, any of their vertices is an element of Q^n and Q^m, respectively.

The next lemma is very important for our considerations. We remark that a polytope is an intersection of a finite number of closed half-spaces. A polytope that has a vertex is called a pointed polytope.

LEMMA 1. *If $L \subseteq R^p_+$ is a nonvoid polytope, then it is a pointed polytope.*

Proof. Suppose that L is not pointed. From theorem 35 [2, Ch. 1] it follows that there exist $x^1 = (x^1_1, \ldots, x^1_n) \in L$ and $x^2 = (x^2_1, \ldots, x^2_n) \in L$, $x^1 \neq x^2$, such that

(1) $$(1 - t)\, x^1 + tx^2 \in L \text{ for all } t \in R.$$

Because $x^1 \neq x^2$, there exists a $j \in \{1, \ldots, n\}$ such that $x^1_j \neq x^2_j$. Taking

$$t^\circ = \begin{cases}(-1 - x^1_j)(x^2_j - x^1_j)^{-1}, & \text{if } x^2_j - x^1_j > 0\\(-1 - x^1_j)(x^2_j - x^1_j)^{-1}, & \text{if } x^2_j - x^1_j < 0\end{cases},$$

we get that $x^\circ = (1 - t^\circ)\, x^1 + t^\circ x^2 \notin L$, since $x^\circ_j < 0$. This contradicts (1). Hence, L is a pointed polytope.

Using lemma 1, we prove an interesting theorem. For future reference denote by (P) respectively by (D) the problems

$$(\mathrm{P}) \begin{cases} f(x) \to \max \\ x \in X, \end{cases} \qquad (\mathrm{D}) \begin{cases} g(x) \to \min. \\ y \in Y \end{cases}$$

THEOREM 2. *The following assertions are true:*

(*i*) *Problem* (P) *is infeasible* (i.e. $X = \Phi$) *if and only if problem* (PQ) is infeasible (i.e. $XQ = \Phi$).

(*ii*) *Problem* (P) *has no optimal solutions* (i.e. $\sup_{x \in X} f(x) = +\infty$) *if and only if problem* (PQ) *has no optimal solutions.*

(*iii*) *Problem* (P) *has optimal solutions if and only if problem* (PQ) *has optimal solutions.*

(*iv*) *If x° is an optimal solution of* (P) *and z° is an optimal solution of* (PQ), *then $f(x^\circ) = f(z^\circ)$.*

Proof. (*i*) If $X = \Phi$, then $XQ = \Phi$, because $XQ \subseteq X$.

Let now $XQ = \Phi$. We suppose that $X \neq \Phi$. Because X is a nonvoid polytope and $X \subseteq R^n_+$, there exists, by virtue of lema la vertex x° of X. However, by remark 1 we get that $x^\circ \in Q^n$. Hence, $x^\circ \in X \cap Q^n = XQ$. This implies that $XQ \neq \Phi$, which is a contradiction. Hence, $X = \Phi$.

(*ii*) If $\sup\{f(x) \mid x \in XQ\} = +\infty$, then $\sup\{f(x) \mid x \in X\} = +\infty$, because $XQ \subseteq X$.

Let now $\sup\{f(x) \mid x \in X\} = +\infty$. Then for each natural number k there exists an element $x^k \in X$ such that $f(x^k) > k$. Because f is a linear function, the set $X_k = \{x \in X \mid f(x) \geqslant k\}$ is for every $k \in N$ a nonvoid polytope. But $X \subseteq R^n_+$. Then $X_k \subseteq R^n_+$ for all $k \in N$. Applying lemma 1, we get that for every $k \in N$ there exists an $z^k \in X$ such that z^k is a vertex of X. By virtue of remark 1 we have $z^k \in Q^n$ for every $k \in N$. Then $z^k \in X \cap Q = XQ$ for every $k \in N$. Now, we have $f(z^k) \geqslant k$ for every $k \in N$, because $z^k \in X_k$. This implies that the function f is not upper bounded on XQ. Hence, $\sup\{f(x) \mid x \in XQ\} = +\infty$.

(*iii*) Because we have proved that (*i*) and (*ii*) are true, it results that problem (P) has optimal solutions if and only if problem (PQ) has optimal solutions.

(*iv*) Let $x^\circ \in X$ be an optimal solution of problem (P) and let $z^\circ \in XQ$ be an optimal solution of problem (PQ). Because $XQ \subseteq X$, we have $f(x^\circ) \geqslant f(z^\circ)$. We prove that the inequality cannot hold.

Suppose that $f(x^\circ) > f(z^\circ)$. Then there exists a rational number t such that $f(x^\circ) > t > f(z^\circ)$. The set $X_t = \{x \in X \mid f(x) \geqslant t\}$ is a nonvoid polytope, because $x^\circ \in X_t$, and $X_t \subseteq R^n_+$ (since $X \subseteq R^n_+$). Applying lemma 1, we get that there exists a vertex z of X_t. Because t and c_j, $j = 1, \ldots, n$, are rational numbers and all vertices of X are elementes of Q^n, we have also $z \in Q^n$. Hence, $z \in X_t \cap Q^n \subseteq X \cap Q^n = XQ$.

Similarly, we can prove:

THEOREM 2′. *The following assertions are true:*

(*i*) *Problem* (D) *is infeasible if and only if problem* (DQ) *is infeasible.*

(*ii*) *Problem* (D) *has no optimal solutions if and only if problem* (DQ) *has no optimal solutions.*

(*iii*) *Problem* (D) *has optimal solutions if and only if problem* (DQ) *has optimal solutions.*

(*iv*) *If y° is an optimal solution of* (D) *and z° is an optimal solution of* (DQ), *then $g(y^\circ) = g(z^\circ)$.*

Using theorems 2 and 2′ and theorem II.8 from [1], we get

THEOREM 3. *For problems* (PQ) *and* (DQ), *one and only one of the following assertions is true:*

(*i*) *both problems have optimal solutions and the optimal values of the objective functions are equal;*

(*ii*) *one of the problems is feasible, while the other is infeasible; in this case, the feasiable problem has no optimal solution;*

(*iii*) *both problems are infeasible.*

REFERENCES

[1] Dragomirescu M., Malița M., *Programare pătratică*, București, Editura Științifică, 1968.

[2] Martos B., *Nonlinear Programming. Theory and Methods*, Budapest, Akademiai Kiadó, 1975.