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A NEW OPERATOR OF BERNSTEIN TYPE

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Abstract. A positive linear operator is considered which generalizes Bernstein's operator. The authors give an easy constructive process and they prove some monotony properties and convexity preserving. Some convergence theorems are given and estimates of the reimander are established.

1. Introduction. It is well-known that to every function f defined on the interval $I=[0,1]$ one can associate the Bernstein polynomial of degree m:

$$(B_m f)(x) = B_m(f(t)\,;\,x) = B_m(\,;\,x) = \sum_{i=0}^{m} p_{m,i}(x)\, f(i/m),$$

where

$$p_{m,i}(x) = \binom{m}{i} x^i (1-x)^{m-i}.$$

It is also well known that B_m is a linear and positive operator and that, for every function f which is continuous on I ($f \in C^\circ(I)$), it results that

$$\lim_{m\to\infty} B_m(f\,;\,x) = f(x)$$

uniformly on I.

In spite of their slow convergence, Bernstein polynomials have been (and they still are) a subject of study by many authors; we can quote above all: Bernstein [3], Berens [2], Lorentz [10], Passow [18], Popoviciu [18], Schoenberg [20], Stancu [21, 22].

A lot of properties of great theoretical and practical interest of the B_m operator have been demonstrated.

This is why in the last twenty years operators of Bernstein type have been constructed and furthermore constructive procedures have been showed. In this respect, we can recall the studies of Baskakov [1], Köning-Zeller [9], Szász [26], and those of Stancu, from which special mention deserve [23, 24, 25].

A complete and updated bibliography on the subject is given in [7].

The properties of i-th iterates of B_m defined by

$$B^i_m(f;\, x) = \sum_{k=0}^{m} \binom{m}{k} B^{i-1}_m(e_k;\, x)/\Delta^k_{1/m} f(0),\quad i \geqslant 1 \tag{1.1}$$

where $e_k(x) = x^k$ and $B^0 = \mathrm{I}$ is the identical operator are also well-known [9, 16].

Suitable linear combinations of the Bernstein polynomials have been studied too; we can recall among them those which are contained in the works of Butzer [4], of Frentiu [6] and of May [11].

Micchelli [15] and Mastroianni-Occorsio [12] introduced and studied, even if separately, the following combinations of iterates of the Bernstein polynomials:

$$B_{m,k}(f;\, x) = \sum_{i=1}^{m} (-1)^{i-1} \binom{k}{i} B^i_m(f;\, x). \tag{1.2}$$

The last one has the property of improving the approximation to increase the approximating function's regularity. Indeed, in the above mentioned works, it has been demonstrated that the polynomial defined by (1.2) approximates a function $f \in C^{2k}(I)$ with a remainder of $O(n^{-k})$ type. It has also been shown in [5] that from all the combinations like

$$\sum_{i=1}^{k} a_{k,i} B^i_m(f;\, x),\quad \sum_{i=1}^{k} a_{k,i} = 1 \tag{1.3}$$

the polynomial (1.2) is the only one to possess the above-mentioned property of convergence.

In [12] Mastroianni and Occorsio remarked that a possible generalization of (1.2) could be

$$B_{m,\lambda}(f;\, x) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{\lambda}{i} B^i_m(f;\, x), \tag{1.3}$$

with λ real positive number.

The aim of this work is to carry out a first study of the $B_{m,\lambda}$ operator which, for $\lambda \in (0,1]$, results positive and preserves many properties of the Bernstein operator.

In § 2, we shall give a constructive procedure of $B_{m,\lambda}$, and in § 3 we shall demonstrate some properties of monotony and of convexity preservation. Finally, in § 4 we shall discuss the convergence of $B_{m,\lambda}$ for $m \to \infty$ and $\lambda \to \infty$ separately, and shall give some estimates of the remainder.

2. The $B_{m,\lambda}$ operator. To every function f defined and limited in I, we associate the continuous function $B_{m,\lambda} f$, $(\lambda \in R^+)$ which is defined through the series:

$$B_{m,\lambda}(f;\, x) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{\lambda}{i} B^i_m(f;\, x), \tag{2.1}$$

where B^i_m is the i-th iterate of the Bernstein operator B_m.

It is easy to verify that, if λ is an integer positive number, (2.1) gives back (1.2) and in particular it results that $B_{m,1} = B_m$. Moreover, for every $\lambda \in R^+$, we have:

$$B_{m,\lambda}(f;\, 0) = f(0);\quad B_{m,\lambda}(f;\, 1) = f(1)$$

$$B_{m,\lambda}(e_i;\, x) = x^i,\quad i = 1,2.$$

We shall give now a procedure to construct $B_{m,\lambda} f$.

To this aim, we denote by S^j_i and s^j_i, respectively, Stirling's numbers of first and second type defined by:

$$x^{(n)} := x\left(x - \frac{1}{m}\right) \dots \left(x - \frac{n-1}{m}\right) = \sum_{j=0}^{n-1} S^n_j \frac{x^{n-j}}{m^j};\quad x^{(0)} = 1,\ x^{(1)} = x;$$

$$x^n = \sum_{j=0}^{n-1} s^n_j \frac{x^{(n-j)}}{m^j}.$$

Now, we set:

$$x_k = (x, x^2, \dots, x^k);\quad u_k = (0, \dots, 0,1)^T \in R^k;$$

$$\Lambda_k = \begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1^{(2)} & 0 \dots 0 \\ \cdot & \cdot & \cdot \quad \cdot \\ 0 & 0 & 0 \dots 1^{(k)} \end{bmatrix};\quad S_k = \begin{bmatrix} 1 & s^2_1/m & s^3_2/m^2 & \dots & s^k_{k-1}/m^{k-1} \\ 0 & 1 & s^3_1/m & \dots & s^k_{k-2}/m^{k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix};$$

then let V_k be the $k \times k$ matrix having for columns the autovectors $v_j(j = \overline{1, k})$ of the matrix $\Lambda_k S_k$, normalized so as to allow that the elements of the principal diagonal of V_k result equal to 1.
It is obvious that

$$\Lambda_k S_k V_k = V_k \Lambda_k.$$

Thence,

$$(\Lambda_k S_k)^j = V_k \Lambda^j_k V^{-1}_k. \tag{2.2}$$

Moreover, let us observe that $V^{-1}_k = U^T_k$, where $(\Lambda_k S_k)^T U_k = U_k \Lambda_k$. Then, it is easy to show (see [8]) that

$$B^{i-1}_m(e_k;\, x) = x_k V_k \Lambda^{i-1}_k V^{-1}_k u_k. \tag{2.3}$$

So, since (2.1), by (1.1) and (2.3), it follows that

$$B_{m,\lambda}(f;\, x) = \sum_{k=0}^{m} \binom{m}{k} x_k V_k \left[\sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i+1} A^i_k \right] V^{-1}_k u_k \Delta^k_{1/m} f(0). \tag{2.4}$$

Now, we observe that

$$\sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i+1} t^i = \frac{1}{t} [1 - (1 - t)^\lambda] =: \gamma(t,\, \lambda);\ t \in (0,1],\ \lambda \in R^+; \tag{2.5}$$

from $1^{(i)} \leq 1$, $(i = \overline{1, k})$, we have:

$$\sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i+1} \Lambda_k^i = G_{k,\lambda},$$

where $G_{k,\lambda}$ is the diagonal matrix whose elements are $\gamma(1^{(i)}, \lambda)$, $i = \overline{1, k}$. In this way (2.4) becomes

$$B_{m,\lambda}(f; x) = \sum_{k=0}^{m} \binom{m}{k} q_{m,k}(x, \lambda)\, \Delta_{1/m}^k f(0), \tag{2.6}$$

where

$$q_{m,k}(x, \lambda) = x_k V_k G_{k,\lambda} V_k^{-1} u_k. \tag{2.7}$$

It is easy to verify that $q_{m,k}(x, \lambda)$ is a polynomial in x of degree not greater than k. So, from (2.7), it follows that $B_{m,\lambda} f$ is a polynomial of degree not greater than m. Furthermore, if p_n is a polynomial of degree not greater than $n \leq m$, the same happens for $B_{m,\lambda} p_n$.

Another representation of $B_{m,\lambda} f$ is

$$B_{m,\lambda}(f; x) = \sum_{k=0}^{m} p_{m,k}(x)\, g(k/m), \tag{2.8}$$

where

$$g(x) = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i+1} B_m^i(f; x); \quad (B_m^0 f = f). \tag{2.9}$$

By easy calculations, we obtain in particular:

$$B_{2,\lambda}(f; x) = f(0) + 2x\, \Delta_{1/2} f(0) + \left[\left(\frac{2}{2^\lambda} - 1\right) x + 2\left(1 - \frac{1}{2^\lambda}\right) x^2\right] \Delta_{1/2}^2 f(0);$$

$$B_{3,\lambda}(f; x) = f(0) + 3x\, \Delta_{1/3} f(0) + 3 \left[\frac{1}{2}\left(-1 + \frac{3}{3^\lambda}\right) x + \right.$$

$$\left. + \frac{3}{2}\left(1 - \frac{1}{3^\lambda}\right) x^2\right] \Delta_{1/3}^2 f(0) + \left[\left(1 - \frac{9}{4}\left(\frac{7^\lambda}{9^\lambda} - \frac{1}{3^\lambda}\right)\right) x - \right.$$

$$\left. - \frac{9}{4}\left(2 + \frac{1}{3^\lambda} - 3\,\frac{7^\lambda}{9^\lambda}\right) x^2 + \frac{9}{2}\left(1 - \frac{7^\lambda}{9^\lambda}\right)\right] x^3 \Delta_{1/3}^3 f(0).$$

Then, let $R_{m,\lambda} = I - B_{m,\lambda}$ be the remainder term, where I is the identical operator. We easily have:

$$R_{m,\lambda} = R_m^\lambda = (I - B_m)^\lambda = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i} B_m^i.$$

from which we deduce that

$$R_{m,\lambda}(e_k; x) = x_k V_k A_k V_k^{-1} u_k, \quad e_k(x) = x^k,$$

where A_k is the diagonal matrix whose elements are: $[1 - 1^{(i)}]^\lambda$.
For example, we have:

$$R_{m,\lambda}(e_2; x) = -B_{m,\lambda}((t - x)^2; x) = -x(1 - x)/m^\lambda \tag{2.10}$$

$$B_{m,\lambda}((t - x^3; x) = -\frac{x(1 - x)^2}{m^\lambda}\left[3 - \left(3 - \frac{2}{m}\right)^\lambda\right]. \tag{2.11}$$

Let us now observe that $B_{m,\lambda}$, $(\lambda \in R^+)$, is not in general a positive operator; it is easy to verify that $B_{m,\lambda}$ is positive if $0 < \lambda < 1$. For these values of λ, $B_{m,\lambda}$ resulkts as a positive operator of Bernstein type.

Finally, note that $B_{m,\lambda}$ with $0 < \lambda < 1$, is a particular case of the operator:

$$R_{m,n}^\lambda f = \sum_{i=n+1}^{\infty} (-1)^{n+1+i} \binom{\lambda}{i} B_m^i f, \quad n < \lambda < n + 1, \quad n \in N.$$

It is not difficult in any case to verify that, for $n > 0$, $R_{m,n}^\lambda f$ does not converge to f. Next, we shall show that $B_{m,\lambda}$ is the only positive operator of the $\{R_{m,n}^\lambda\}_{n \in N}$ class converging to f.

3. On the properties of $B_{m,\lambda}$ $(0 < \lambda < 1)$. The following propositions are true.

PROPOSITION 3.I *The operator $B_{m,\lambda}$ keeps the convexity (concavity) of every order of the function f.*

Indeed, from (2.8) we deduce that

$$\frac{(B_{m,\lambda} f)^{(p)}(x)}{p!} = 1^{(p)} \sum_{k=0}^{m-p} p_{m-p,k}(x) \left[\frac{k}{m}, \frac{k+1}{m}, \ldots, \frac{k+p}{m}; g\right], \quad p < m.$$

Moreover, if f is convex of order $p - 1$, by (2.9) it follows that the same happens for g; so the assertion is proven.

PROPOSITION 3.II *For every $x \in (0,1)$, the relation*

$$B_{m+1,\lambda}(f; x) - B_{m,\lambda}(f; x) = -x(1 - x)[x_1, x_2, x_3; f]\left(\frac{1}{m^\lambda} - \frac{1}{(m+1)^\lambda}\right)$$

holds, where x_i, $(i = 1, 2, 3)$, are three suitable points of I.

Indeed, by (2.8) we have

$$A_m(f; x) := B_{m+1,\lambda}(f; x) - B_{m,\lambda}(f; x) =$$

$$= -\frac{x(1 - x)}{m(m+1)} \sum_{k=0}^{m-1} p_{m-1,k}(x) \left[\frac{k}{m}, \frac{k+1}{m}, \frac{k+1}{m+1}; g\right]$$

So, the operator A_m has degree of exactness 1.

Furthermore, if f is convex of first order, g will be also so; in this case, we have $A_m(f; x) \neq 0$ for every $x \in (0,1)$. For Popoviciu's theorem [19], three points $x_1, x_2, x_3 \in (0,1)$ exist such that

$$A_m(f; x) = A_m(e_2; x)\,[x_1, x_2, x_3; f].$$

Then, from (2.8) it follows that

$$A_m(e_2; x) = -x(1-x)\left[\frac{1}{m^\lambda} - \frac{1}{(m+1)^\lambda}\right],$$

and so the assertion is proven.
Moreover, from Proposition 3.II, we may claim

COROLLARY 3.III. *If the function f is convex of the first order, then, for every $x \in (0,1)$, it results that*

$$B_{m+1,\lambda}(f; x) < B_{m,\lambda}(f; x).$$

Since $B_{m,\lambda}$ has degree of exactness 1 and by a theorem which has been proved in [13], the following representation of the remainder

$$R_{m,\lambda}(f; x) = -\frac{x(1-x)}{2m^\lambda} f''(\xi), \quad \xi \in (0,1), \tag{3.1}$$

holds; by (3.1), the other one can be deduced:

$$R_{m,\lambda}(f; x) = -\frac{x(1-x)}{m^\lambda}\,[\bar{x}_1, \bar{x}_2, \bar{x}_3; f] \tag{3.2}$$

$\bar{x}_1, \bar{x}_2, \bar{x}_3$ being suitable points of (0.1).

4. On the convergence of $B_{m,\lambda}$. First of all, let us suppose that $0 < \lambda < 1$. In this case, $B_{m,\lambda}$ verifies Korovkin's conditions, and consequently the sequence $\{B_{m,\lambda} f\}$ converges uniformly to f on I. For some classes of functions, a measure of approximation is given by

THEOREM 4.1. *For every continuous function f and every $\lambda \in (0,1)$ it results that*

$$\|f - B_{m,\lambda} f\| \leqslant \frac{5}{4}\,\omega(f; 1/m^{\lambda/2}); \tag{4.1}$$

$$\|f - B_{m,\lambda} f\| \leqslant \frac{3}{4}\,\frac{\omega(f'; 1/m^{\lambda/2})}{m^{\lambda/2}}, \quad f' \in C^0(I), \tag{4.2}$$

where the norm is the uniform one.

To prove (4.1), we observe that if $\omega(f;)$ denotes the modulus of continuity of the function f, it results that

$$\omega(f; |x-t|) \leqslant \left(1 + \frac{(x-t)^2}{\delta^2}\right)\omega(f; \delta),$$

for every x, $t \in I$ and for every $\delta > 0$

So, we have:

$$|f(x) - B_{m,\lambda}(f; x)| \leqslant B_{m,\lambda}(|f(x) - f(t)|; x) \leqslant$$

$$\leqslant B_{m,\lambda}(\omega(f; |t-x|; x) \leqslant \left(1 + \frac{B_{m,\lambda}((t-x)^2; x)}{\delta^2}\right)\omega(f; \delta).$$

Hence, by (2.11), we deduce that

$$|f(x) - B_{m,\lambda}(f; x)| \leqslant \left(1 + \frac{x(1-x)}{m^\lambda \delta^2}\right)\omega(f; \delta), \quad \delta > 0.$$

In particular, for $\delta = m^{-\lambda/2}$, (4.1) follows.
To prove (4.2), we observe that

$$-f(x) + f(t) = f'(x)(t-x) + \int_x^t [f'(u) - f'(x)]\,du =: f'(x)(t-x) + F(t, x), \tag{4.4}$$

$$|F(t, x)| \leqslant |t-x|\,\omega(f'; |t-x|) \leqslant \left(|x-t| + \frac{(x-t)^2}{\delta}\right)\omega(f; \delta), \quad \delta > 0, \tag{4.5}$$

for every $x, t \in I$.
So, we have

$$|f(x) - B_{m,\lambda}(f; x)| = |B_{m,\lambda}(F(t, x); x)| \leqslant$$

$$\leqslant \sqrt{\frac{x(1-x)}{m^\lambda}}\left(1 + \frac{1}{\delta}\sqrt{\frac{x(1-x)}{m^\lambda}}\right)\omega(f'; \delta);$$

from which, for $\delta = m^{-\lambda/2}$, we have (4.2).

The foregoing inequalities give back, for $\lambda = 1$, the well-known relations corresponding to the Bernstein polynomials.

Now, let $\lambda > 1$. The case of λ as an integer number greater than 1 has already been studied in detail in [12]; so, let us suppose that $\lambda = i + \gamma$, where i is the maximum integer number such that $k \leqslant \lambda$ and $\gamma \in (0,1)$.

THEOREM 4.II. *Let $\lambda = k + \gamma$ where k is an integer number and $\lambda \in (0,1)$. If $f \in C^{2k+2}(I)$, then the following inequality*

$$\|f - B_{m,\lambda} f\| \leqslant \frac{c\|f\|_{2k}}{m^{k+\gamma}},$$

holds, where $\|f\|_{2k} = \max\limits_{2 \leqslant i \leqslant 2k} \|f^{(i)}\|$ and c is a constant depending on k and independent of f and m.

In fact, by (3.10), we have

$$|R_{m,\lambda}(f; x)| = |R_{m,\gamma}(R_{m,k} f; x)| \leqslant \frac{\|(R_{m,k} f)''\|}{8m^\gamma}.$$

On the other side, if $f \in C^{2k+2}(I)$, it results that (see [14], Th. 5.1)

$$\|(R_{m,k})''\| \leqslant \frac{c}{m^k} \|f\|_{2k},$$

and the assertion is proven.
Finally, we prove

THEOREM 4.III. *If f is defined on I, then the relation*

$$\lim_{\lambda\to\infty} B_{m,\lambda}(f; x) = L_m(f; x)$$

holds, where $L_m(f; x)$ is the Lagrange interpolating polynomial corresponding to the function f and to the knots $\{i/m\}$, $i = 0, m$.

In order to prove the theorem, we observe that, if we substitute s_i^j by Stirling's numbers of first order S_i^j, in the elements of the matrix, S_k, we obtain the matrix $\mathscr{S}_k = S_k^{-1}$.

Now, by (2.6) we obtain

$$\lim_{\lambda\to\infty} B_{m,\lambda}(f; x) = \sum_{k=0}^{m} \binom{m}{k} [\lim_{\lambda\to\infty} q_{m,k}(x, \lambda)] \Delta_{1/m}^k f(0),$$

and from (2.7),

$$\lim_{\lambda\to\infty} B_{m,\lambda}(f; x) = \sum_{k=0}^{m} \binom{m}{k} x_k V_k (\lim_{\lambda\to\infty} G_{m,\lambda}) V_k^{-1} u_k \Delta_{1/m}^k f(0).$$

Furthermore, since $0 \leqslant (1 - 1^{(i)}) < 1$, by (2.5), we deduce that

$$\lim_{\lambda\to\infty} G_{k,\lambda} = \Lambda_k^{-1}.$$

Hence, by (2.2), it results that

$$\lim_{\lambda\to\infty} B_{m,\lambda}(f; x) = \sum_{k=0}^{m} \binom{m}{k} x_k (\Lambda_k S_k)^{-1} u_k \Delta_{1/m}^k f(0). \tag{4.6}$$

Since

$$x_k (\Lambda_k S_k)^{-1} u_k = x_k \mathscr{S}_k \Lambda_k^{-1} u_k = \frac{x^{(k)}}{1^{(k)}},$$

and from (4.6), the assertion is immediate.

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