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BERNSTEIN'S POLYNOMIALS FOR POWERS *VIA* SHIFTING OPERATOR

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1. In what follows we shall consider the sequence of Bernstein's polynomial operators $(B_n)_{n\geqslant 1}$, $B_n : C[0, 1] \to C[0, 1]$ defined by

$$B_n(f\,;\,x) = \sum_{i=0}^{n} p_{n,i}(x)\,f\left(\frac{i}{n}\right), \tag{1}$$

where $p_{n,i}(x) = \binom{n}{i} x^i(1-x)^{n-1}$ $i = 0, 1, \ldots, n$ are Bernstein's basic polynomials.

Let $e_s \in C[0, 1]$, $e_s(x) = x^s$, $s = 0, 1, \ldots$. In many problems in connection with the approximation of continuous functions by Bernstein's polynomials it is important to write the expression of $B_n(e_s\,;\,x)$ in a suitable form (see, for example, [5]). An algebraic method for the calculation of $B_n(e_s\,;\,x)$ was given recently by HE in [4]. Using the technique of generating functions, we determined in [1] the expression of $B_n(e_s\,;\,x)$ and established some combinatorial properties.

The aim of this paper is to calculate $B_n(e_s\,;\,x)$ using CHANG's interesting idea [2] based on writing Bernstein's polynomials in terms of the shifting operator.

2. We begin with some properties in connection with the expression of Bernstein's polynomials in terms of the operators I, E and Δ which appear in the finite difference theory. These results are given in [2].

Let $f \in C[0, 1]$. We denote $f_i = f(i/n)$, $i = 0, 1, \ldots, n$. The operators I, E, Δ are given by $If_i = f_i$, $Ef_i = f_{i+1}$ and $\Delta = E - I$. Using these three operators, we observe that the following equality holds :

$$B_n(f\,;\,x) = [(1-x)I + xE]^n f. \tag{2}$$

But $(1-x)I + xE = x(E - I) + I = x\Delta + I$ and there follows that relation (2) becomes

$$B_n(f\,;\,x) = (I + x\Delta)^n f. \tag{3}$$

Since the operators I and Δ commute, applying the binomial formula, from (3) we get

$$B_n(f\,;x) = \sum_{i=0}^{n} \binom{n}{i} \Delta^i f_0\, x^i. \tag{4}$$

We recall that (see [3] pp. 34)

$$\Delta^i f_0 = \sum_{\nu=0}^{i} (-1)^{i-\nu} \binom{i}{\nu} f_\nu \tag{5}$$

Now, using the relations (4) and (5), we can easily compute $B_n(e_s\,;x)$. Let us suppose that $f(x) = e_s(x) = x^s$, $x \in [0, 1]$. Then $f_\nu = \nu^{s/ns}$, $\nu = 0, 1, \ldots, n$. In this case, from (5) it results that

$$\Delta^i f_0 = \frac{1}{n^s} \sum_{\nu=0}^{i} (-1)^{i-\nu} \binom{i}{\nu} \nu^s. \tag{6}$$

Using the identity (see [7], Problem 189, p. 42)

$$\sum_{l=0}^{i} (-1)^l \binom{i}{1} (i-1)^s = i!\, S_i^s, \tag{7}$$

where S_i^s are Stirling's numbers of the second kind, from (4), (5) and (6), we obtain

$$B_n(e_s\,;x) = \sum_{i=0}^{n} \binom{n}{i} \frac{i!}{n^s} S_i^s x^i.$$

REFERENCES

[1] Andrica, D., *Powers by Bernstein's operators and some combinatorial properties*, Itinerant seminar on functional equations, approximation and convexity, Cluj-Napoca, Preprint No. 6 (1985), pp. 5–9.
[2] Chang, G.-Z., *Bernstein polynomials via the shifting operator*. The American Mathematical Monthly, **91** (1984), nr. *10*, 634–638.
[3] Ghelfond, A., O., *Calculul cu diferențe finite*, Edit. Tehnică, București, 1956.
[4] He, F., *The powers and their Bernstein polynomials*, Real Analysis Exchange, **9** (1983–1984), no. *2*, 578–583.
[5] Karlin, S., Ziegler, Z., *Iteration of Positive Approximation Operators*, J. Approx. Theory, **3** (1970), 310–339.
[6] Lorentz, G., G., *Bernstein Polynomials*, University of Toronto Press, Toronto 1956.
[7] Polya, G., Szegő, G., *Problems and Theorems in Analysis*, Vol. I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

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