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NUMERICAL SOLUTION OF SINGULARLY PERTURBED BOUNDARY-VALUE PROBLEMS USING ADAPTIVE SPLINE FUNCTION APPROXIMATION

K. SURLA
(Novi Sad)

Abstract. We prove that an exponential fitting cubic spline difference scheme, when applied with a uniform mesh of size h to: $\varepsilon y'' + p(x)y' = f(x)$ for $0 < x < 1$, $p(x) \geqslant p > 0$, $u(0)$ and $u(1)$ given, is uniformly convergent (i.e. the error is bounded by $Mh^2/(\varepsilon + h)$ with the constant M independent of h and ε. The convergence between the grid points is considered. This result is ilustrated by numerical experiments.

AMS Subjects Classifications: 65L10

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1. Introduction and notation. We wish to introduce the concept of employing cubic spline collocation with "artifical viscosity" in the approximation of the singularly perturbed problem:

$$(1.1)\qquad \begin{aligned} &Ly = \varepsilon y'' + p(x)y' + q(x)y = f(x),\ 0 < x < 1 \\ &y(0) = \delta_0,\ y(1) = \delta_1 \end{aligned}$$

where ε is a parameter in $(0, 1]$, δ_0 and δ_1 are given constants, $p, q, f \in C^3[0, 1]$, $p(x) \geqslant \bar{p} > 0$, $q(x) \leqslant 0$. Under these assumptions, (1.1) has a unique solution y with a boundary layer at $x = 0$ for "small" ε.

Classical cubic spline collocation methods when applied to (1.1) on a uniform grid have an inherent formal cell Reynolds number limitation. This leads to the spurious oscillations or gross inaccuracies in the approximate solution ([1], [4], [5], [7]). In order to avoid these difficulties one introduces exponential functions into the spline basis (spline in tension [7], γ-elliptic splines and γ_ε-splines [4], adaptive splines [11], finite elements [3], [13]). In this paper, the exponential features of the exact solution are transferred to spline coefficients by the "artifical viscosity" $\sigma_j = (hp_j/2)\mathrm{cth}(hp_j/(2\varepsilon))$. In that way, a spline difference scheme (2.7) is obtained. In the case $p(x) = p = \mathrm{const}$, $q(x) = 0$, its local truncation error for the functions 1, x, $\exp(-xp/\varepsilon)$ is equal to zero. In the same case the spline in tension [7] is a linear combination of the four functions 1, x, $\exp(-xp/\varepsilon)$, $\exp(xp/\varepsilon)$ on each subinterval.

The spline collocation method of this type is derived in [15] based on the analysis given in [10] and [9]. An outline of the proof of Theorem 1 is given in [16]. Here, we consider an approximate solution of (1.1) in the case $q(x) = 0$, in the form of the cubic spline function $v(x) \in C^2[0, 1]$. The obtained estimate for the $z(x) = y(x) - v(x)$ shows the uniform convergence of the first order in ε at the grid points.

If ε is fixed, the obtained estimates give the result from [15] which is the same as in [10] when $\sigma_j = \varepsilon$. The numerical results suggest that, when $q(x) \neq 0$, (2.7) also achieves uniform first-order accuracy.

Let the grid points $\{x_j\}$ be given by $x_j = j \cdot h$, $j = 0(1)n+1$, $h = 1/(n+1)$. The spline function $v(x)$ on the interval $[x_j, x_{j+1}]$ has the form

$$(1.2)\qquad v(x) = v_j^{(0)} + v_j^{(1)}(x - x_j) + \frac{v_j^{(2)}}{2\,!}(x - x_j)^2 + \frac{v_j^{(3)}}{3\,!}(x - x_j)^3,$$

where $v_j^{(k)}$ $(k = 0, 1, 2, 3)$ are constants to be determined. The spline difference scheme which we are going to derive has the form

$$(1.3)\qquad \begin{cases} r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1} = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}, \; j = 1(1)n \\ v_0 = \delta_0, \; v_{n+1} = \delta_1, \text{ where} \\ v_j = v_j^{(0)} = v(x_j), \; f(x_j) = f_j. \end{cases}$$

For an arbitrary mesh function $\{g_j\}$ we introduce the notation

$$(1.4)\qquad R_h g_j = r_j^- g_{j-1} + r_j^c g_j + r^+ g_{j+1}$$

$$(1.5)\qquad Q_h g_j = q_j^- g_{j-1} + q_j^c g_j + q_j^+ g_{j+1}$$

The local truncation error $\tau_j(g)$ of scheme (1.3) for the arbitrary function $g(x) \in C^2[0, 1]$ is defined by

$$(1.6)\qquad \tau_j(g) = R_h(g(x_j)) - Q_h(Lg(x_j))$$

When it is clear from the context the j subscripts in $r_j^-, \ldots, q_j^+$ will be omitted.

Throughout the paper M and M_i will be used to denote possibly different positive constants independent of ε and h.

2. Derivation of the method and some properties. We want to obtain the solution of the problem (1.1) in the form of the cubic spline (1.2).

The coefficients $v_j^{(k)}$, $(k = 0, 1, 2, 3)$ are determined from the equations

$$(2.1)\qquad \sigma_j v_j^{(2)} + p_j v_j^{(1)} + q_j v_j^{(0)} = f_j, \; j = 0(1)n$$

$$(2.2)\qquad \sigma_{n+1}(v_n^{(2)} + hv_n^{(3)}) + p_{n+1}(v_n^{(1)} + hv_n^{(2)} + h^2 v_n^{(3)}/2) + q_{n+1}(v_n^{(0)} + hv_n^{(1)} + h^2 v_n^{(2)}/2 + h^3 v_n^{(3)}/6) = f_{n+1}$$

$$\sigma_j = \varepsilon\rho_j \operatorname{cth}\rho_j, \quad \rho_j = hp_j/(2\varepsilon), \quad p_j = p(x_j), \quad q_j = q(x_j)$$

$$(2.3)\qquad v_j^{(0)} = v_{j-1}^{(0)} + hv_{j-1}^{(1)} + h^2 v_{j-1}^{(2)}/2 + h^3 v_{j-1}^{(3)}/6, \qquad j = 1(1)n$$

$$(2.4)\qquad v_j^{(1)} = v_{j-1}^{(1)} + hv_{j-1}^{(2)} + h^2 v_{j-1}^{(3)}/2, \qquad j = 1(1)n$$

$$(2.5)\qquad v_j^{(2)} = v_{j-1}^{(2)} + hv_{j-1}^{(3)}, \qquad j = 1(1)n$$

$$(2.6)\qquad v_0^{(0)} = \delta_0, \quad v_{n+1}^{(0)} = \delta_1$$

Equations (2.1) and (2.2) are introduced instead of the collocations equations. Equalities (2.3)—(2.6) result from the requirement that $v(x) \in C^2[0, 1]$ and from the boundary conditions. By the elimination of $v_j^{(1)}$, $v_j^{(2)}$ and $v_j^{(3)}$ from the above equations as in [10], we obtain the scheme:

$$(2.7)\qquad R_h v_j = Q_h f_j, \; j = 1(1)n, \text{ where}$$

$$r_j^- = \beta_j b_j/\gamma_j + hq_{j-1}/(2\sigma_{j-1}), \quad r_j^+ = a_j\bar{\alpha}_{j+1}/\gamma_{j+1}$$

$$r_j^c = -a_j\beta_{j+1}/\gamma_{j+1} - b_j\bar{\alpha}_j/\gamma_j + hq_j/(2\sigma_j),$$

$$a_j = 1 + 1/\omega_j, \quad b_j = 1 - 1/\omega_{j-1}, \quad \omega_j = \operatorname{cth}\rho_j,$$

$$\bar{\alpha}_j = 1 + h^2 q_j/(6a_j\sigma_j), \quad \beta_j = 1 - h^2 q_{j-1}(4a_j\sigma_j - hp_j)/(12a_j\sigma_j\sigma_{j-1})$$

$$\gamma_j^{-1} = 3\omega_{j-1}(\omega_j + 1)/(hA_{j-1}), \quad A_j = 3\omega_{j+1} + 2\omega_j - 2\omega_{j+1} - 1$$

$$q_j^- = \frac{h^2 b_j}{6\gamma_j}\left(\frac{p_j h}{2\sigma_{j-1}\sigma_j a_j} - \frac{2}{\sigma_{j-1}}\right) + \frac{h}{2\sigma_{j-1}}$$

$$q_j^c = \frac{h^2 b_j}{6\gamma_j}\left(\frac{(p_j h}{2\sigma_j^2 a_j} - \frac{1}{\sigma_j}\right) - \frac{h^2 a_j}{6\gamma_{j+1}}\left(\frac{p_{j+1}h}{2\sigma_{j+1}\sigma_j a_{j+1}} - \frac{2}{\sigma_j}\right) + \frac{h}{2\sigma_j}$$

$$q_j^+ = \frac{h^2 a_j}{6\gamma_{j+1}}\left(\frac{1}{\sigma_{j+1}} - \frac{p_{j+1}h}{2\sigma_{j+1}^2 a_{j+1}}\right)$$

$$v_0 = \delta_0, \quad v_{n+1} = \delta_1$$

For $q(x) \equiv 0$, this scheme reduces to

$$(2.8)\qquad \begin{cases} r_j^- = 3(\omega_{j-1} - 1)/(hA_{j-1}), \; r_j^+ = 3(\omega_{j+1} + 1)/(hA_j) \\ r_j^c = -r_j^- - r_j^+ \\ q_j^- = 1/(p_{j-1}A_{j-1}), \; q_j^+ = 1/(p_{j+1}A_j) \\ q_j^c = ((2\omega_{j-1} - 1)/A_{j-1} + (2\omega_{j+1} + 1)/A_j)/(\omega_j p_j) \end{cases}$$

LEMMA 1. (maximum principle). *Let $\{V_j\}$ be a set of values at the grid points x_j satisfying $V_0 \leq 0$, $V_{n+1} \leq 0$ and $R_h V_j \geq 0$, $j = 1(1)n$. Then $V_j \leq 0$ for $j = 0(1)n+1$*

Proof. See [1], Remark 2.2.

LEMMA 2. ([2]). *Let* $f,\ p \in C^3[0,1]$. *Then the solution of* (1.1) *can be written in the form*

$$y(x) = (-\varepsilon y'(0)/p(0))\exp(-p(0)x/\varepsilon) + w(x), \text{ where}$$

$$|w^{(i)}(x)| \leqslant M(1+\varepsilon^{-i+1}\exp(-2\delta x/\varepsilon)),\quad i = 0(1)4, \tag{2.9}$$

M and δ are constants independent of h and ε.

LEMMA 3 ([12]). *Equation* (1.1) *has a unique solution and there are positive constants δ and M, independent of ε, such that*

$$|y^{(i)}(x)| \leqslant M + M\varepsilon^{-i}\exp(-2\delta x/\varepsilon),\ i = 0(1)4. \tag{2.10}$$

As in [2] we use two comparison functions $\Phi_j = -2 + x_j$, $\psi_j = -\exp(-\beta x_j/\varepsilon)$ for some $\beta > 0$ to be chosen. From Lemmas 2 and 3 we can conclude that

$$|y(x)| \leqslant M\exp(-p(0)x/\varepsilon) + |w(x)|)$$

and because of that we consider $R_h\Phi_j$ and $R_h\psi_j$.

At the beginning, we introduce some relations and some estimates for hyperbolic functions.

$$|\operatorname{cth} t - 1| \leqslant Mt^2/(1+t),\ t \geqslant 0, \tag{2.11}$$

$$\operatorname{cth}\frac{ah}{2\varepsilon} - \operatorname{cth}\frac{p_jh}{2\varepsilon} = \operatorname{sh}\frac{(p_j-a)}{2\varepsilon}h\Big/\left(\operatorname{sh}\frac{ah}{2\varepsilon}\cdot\operatorname{sh}\frac{p_jh}{2\varepsilon}\right) \tag{2.12}$$

$$Mt \leqslant \operatorname{sh} t \leqslant M_1 t,\ 0 \leqslant t \leqslant M_2 \tag{2.13}$$

$$Me^t \leqslant \operatorname{sh} t \leqslant M_1 e^t,\ M_2 \leqslant t \leqslant \infty \tag{2.14}$$

These estimates are taken from [12].

$$\omega_{j+1} - \omega_j = -\frac{h^2}{2\varepsilon}p'(\xi)\frac{1}{\operatorname{sh}^2(\eta)}, \tag{2.15}$$

η is a point between ρ_{j+1} and ρ_j, $x_j \leqslant \xi \leqslant x_{j+1}$.

$$|A_j - A_{j-1}| \leqslant \begin{cases} M\dfrac{\varepsilon^3}{h^3}\operatorname{sh}\left(\dfrac{h^2}{2\varepsilon}p'(\xi)\right) & \text{when } h \leqslant \varepsilon,\ x_{j-1} \leqslant \xi \leqslant x_{j+1} \\ M\dfrac{h^2}{\varepsilon}p'(\xi)e^{-2\bar\eta} & \text{when } \varepsilon \leqslant h \end{cases} \tag{2.16}$$

$\bar\eta$ is a point between the points ρ_{j-1} and ρ_{j+1}.

$$|\omega_{j+1} - 2\omega_j + \omega_{j-1}| \leqslant \begin{cases} Mh\varepsilon & \text{for } h \leqslant \varepsilon \\ Mh^2\exp(-h\beta/\varepsilon) & \text{for } \varepsilon \leqslant h \end{cases} \tag{2.17}$$

$$r^- = r^-(\rho_{j-1},\rho_j) = r^-(\rho_j,\rho_j) + (\rho_{j-1}-\rho_j)D_{r^-}(\rho_j,\rho_j) + (\rho_{j-1}-\rho_j)^2 D^2_{r^-}(\bar\eta_1,\rho_j)/2, \tag{2.18}$$

where $\bar\eta_1$ is a point between ρ_{i-1} and ρ_i, and

$$D^i_r = \frac{\partial^i r(z,\rho_j)}{\partial z^i},\quad D^i_r(a,b) \text{ is a value of } D^i_r \text{ at the point } (a,b).$$

$$D_{r^-}(\rho_j,\rho_j) = -\frac{3}{h}\frac{(\omega_j^2-1)(\omega_j+1)}{(3\omega_j^2-1)^2} \tag{2.19}$$

$$r^+ = r^+(\rho_{j+1},\rho_j) = r^+(\rho_j,\rho_j) + (\rho_{j+1}-\rho_j)D_{r^+}(\rho_j,\rho_j) + (\rho_{j+1}-\rho_j)^2 D^2_{r^+}(\eta,\rho_j)/2, \tag{2.20}$$

η is a point between ρ_j and ρ_{j+1}.

$$D_{r^+}(\rho_j,\rho_j) = \frac{3}{h}\frac{(\omega_j^2-1)(\omega_j-1)}{(3\omega_j^2-1)^2} \tag{2.21}$$

$$D_{r^-} - D_{r^+} = -\frac{6}{h}\frac{(\omega_j^2-1)\omega_j}{(3\omega_j^2-1)^2} \tag{2.22}$$

$$r^-(\rho_j,\rho_j)/r^+(\rho_j,\rho_j) = \exp(-hp_j/\varepsilon) \tag{2.23}$$

$$\frac{r^-}{r^+} = \exp(-hp_j/\varepsilon) - r^-(\rho_j,\rho_j)R_j/(r^+(\rho_j,\rho_j))^2 + C_j/r^+(\rho_{j+1},\rho_j) + O(h^2),\ h \leqslant M\varepsilon, \tag{2.24}$$

$$R_j = (\rho_{j-1}-\rho_j)D_{r^-}(\bar\eta_1,\rho_j),\quad C_j = (\rho_{j+1}-\rho_j)D_{r^+}(\eta,\rho_j),$$

$$\frac{\partial q^-}{\partial\rho_{j-1}} = (\omega_{j-1}^2-1)(3\omega_j+2)/(p_{j-1}A_{j-1}^2) \tag{2.25}$$

$$\frac{\partial q^+}{\partial\rho_{j+1}} = (\omega_{j+1}^2-1)(3\omega_j-2)/(P_{j+1}A_j^2) \tag{2.26}$$

$$\frac{\partial q^c}{\partial\rho_{j+1}} = (\omega_{j+1}^2-1)(A_j+\omega_j)/(p_j\omega_jA_j^2) \tag{2.27}$$

$$\frac{\partial q^c}{\partial\rho_{j-1}} = (\omega_{j-1}^2-1)(A_{j-1}-\omega_j)/(p_j\omega_jA_{j-1}^2) \tag{2.28}$$

$$e^{-\frac{\delta x_j}{\varepsilon}} \leqslant M(\varepsilon^k/h^k)\exp(-\delta x_j/2\varepsilon),\ \delta > 0,\ j \neq 0. \tag{2.29}$$

LEMMA 4. *There are constants c_1 and c_2 independent of h and ε, such that for $h \leqslant c_1$, $0 < \beta \leqslant c_2$, $j = 1(1)n$,*

a) $R_h\varphi_j \geq M\dfrac{h^2}{\varepsilon^2}$ *for* $h \leqslant \varepsilon$

b) $R_h\varphi_j \geq M$ *for* $\varepsilon \leqslant h$,

c) $R_h\psi_j \geq M\mu^j(\beta)/h$ for $h \geqslant \varepsilon$,

d) $R_h\psi_j/\mu(\beta) \geqslant M\mu^{j-1}(\beta)/h$ for $h \geqslant \varepsilon$

e) $R_h\psi_j \geq M\dfrac{h^2}{\varepsilon^2}\dfrac{1}{\varepsilon}\mu^j(\beta)$ for $h \leqslant \varepsilon$.

$\mu(\beta) = \exp(-\beta h/\varepsilon)$

Proof. a) $R_h\varphi_j = -r^-h + r^+h = 3\left(\dfrac{\omega_{j+1}+1}{A_j} - \dfrac{\omega_{j-1}-1}{A_{j-1}}\right) \geq$

$\geq M\dfrac{h^2}{\varepsilon^2}$ because of (2.12), (2.13), (2.16).

b) $R_h\varphi_j \geq M_1(1/A_j + 1/A_{j-1}) \geq M$

c) $R_h\psi_j = \mu(\beta)^{j-1}r^+(1-\mu(\beta))\left(\mu(\beta) - \dfrac{r^-}{r^+}\right)$

Let $\varepsilon \leqslant hc_3$, then $r^+ \geq M_1/h$

$$r^-/r^+ = E_j(\omega_{j-1}-1)\,;\; E_j = \frac{A_j}{A_{j-1}(\omega_{j+1}+1)},\quad \lim_{\varepsilon\to 0} E_j = \frac{1}{2},$$

$$\omega_{j-1} - 1 = 2\exp(-2\rho_{j-1})/(1-\exp(-2\rho_{j-1})) \leqslant M_2\exp(-2\rho_{j-1}),$$

$$1 - \mu(\beta) \geq M_3 \text{ and } \mu(\beta) - \frac{r^-}{r^+} \geqslant M_4\exp(-\beta h/\varepsilon)$$

for appropriately chosen c_2 and c_3. Thus, c) follows for $\varepsilon \leqslant hc_3$.

e) $r^+ \geq M/\varepsilon$, $1-\mu(\beta)\dfrac{h}{\varepsilon}\exp(-\theta h\beta/\varepsilon)$, $0 < \theta < 1$,

$\varepsilon \geq C_3h$

$$\mu(\beta) - \frac{r^-}{r^+} \geq M[\exp(-h\beta/\varepsilon) - \exp(-hp_j/\varepsilon) +$$

$$+ r^-(\rho_j, \rho_j)R_j/(r^+(\rho_j, \rho_j))^2 +$$

$$+ C_j/r^+(\rho_{j+1}, \rho_j)] \geq M\frac{h}{\varepsilon}\exp(-\beta h/\varepsilon) \text{ (see (2.19), (2.21), (2.24))}.$$

Thus,

$R_h\psi_j \geq M\mu^j(\beta)\,h^2\varepsilon^{-3}$ and e) is true. For $c_3 \leqslant \varepsilon/h < 1$ from the last inequality we obtain c).

d) By division of the left and right side of the inequality c) with $\mu(\beta)$ we can conclude that d) is true.

3. Proof of the uniform convergence at the grid points. The truncation error of the scheme (2.8) is defined to be

(3.1) $$\tau_j = R_h\,y(x_j) - Q_h\,(Ly(x_j)) = R_h\,y(x_j) - R_h\,v_j = R_hz_j,$$

$$z_j = y(x_j) - v_j$$

According to [1], [2] we use the following form of the τ_j

$$\tau_j = \tau_j(y) = T_j^0\,y(x_j) + T_j^1\,y'(x_j) + \cdots$$

$$T_j^0 = r_j^+ + r_j^c + r_j^-$$

$$T_j^1 = h(r_j^+ - r_j^-) - (q_j^+p_{j+1} + q_j^c\,p_j + q_j^-\,p_{j-1})$$

$$T_j^2 = h^2(r_j^+ + r_j^-)/2 - \varepsilon(q_j^+ + q_j^c + q_j^-) - h(q_j^+\,p_{j+1} - q_j^-\,p_{j-1})$$

$$T_j^3 = h^3(r_j^+ - r_j^-)/6 + (\varepsilon h - p_{j-1}h^2/2)q_j^- - (p_{j+1}h^{-2}/2 + \varepsilon h)q_j^+.$$

Expanding out to $y^{(4)}(x)$ terms, we have ($T_j^0 = 0$, $T_j^1 = 0$):

(3.2) $$\tau_j = T_j^2y^{(2)}(x_j) + T_j^3\,y^{(3)}(x_j) + r^-\,R_3(x_j, x_j - h, y) +$$
$$+ r^+\,R_3(x_j, x_j + h, y) - q^-\,\varepsilon R_1(x_j, x_j - h, y'') -$$
$$- q^-\,p_{j-1}\,R_2(x_j, x_j - h, y') - \varepsilon q^+\,R_1(x_j, x_j + h, y'') -$$
$$- q^+\,(p_{j+1}R_2(x_j, x_j + h, y'),\ \text{where}$$

(3.3) $$R_k(a,b,\,y) = g^{(k+1)}(\xi)\frac{(b-a)^{k+1}}{(k+1)\,!} = \frac{1}{k\,!}\int_a^b (b-s)^k\,g^{(k+1)}\,(s)ds,$$

ξ is a point between the points a and b. We shall use the above expression when $h \leqslant \varepsilon$, but when $\varepsilon \leqslant h$ we shall use

(3.4) $$\tau_j = T^2y^{(2)}(x_j) + r^-\,R_2(x_j, x_j - h, y) + r^+\,R_2(x_j, x_j + h, y)$$
$$- q^-\,\varepsilon R_0(x_j, x_j - h, y'') - \varepsilon q^+\,R_0(x_j, x_j + y, y'')$$
$$- q^-\,p_{j-1}R_1(x_j, x_j - h,\ y') - q^+\,p_{j+1}\,R_1(x_j, x_j + h,\ y')$$

and the remainder terms in the integral form.

The technique of the proof is outlined in [2] for El-Mistikawy and Werle's scheme.

It is based on the comparison function approach of Kellogg and Tsan [12], i.e. on the following corollary of the maximum principle.

COROLLARY 1. *If* $k_1(h,\ \varepsilon) \geqslant 0$ *and* $k_2(h,\ \varepsilon) \geqslant 0$ *are such functions that* $R_h(k_1\varphi_j + k_2\psi_j) \geq R_h(\pm z_j) = \pm\,\tau_j$ *for* $j = 1(1)n$, *then* $|z_j| \leqslant k_1|\varphi_j| + k_2|\psi_j|$. *So, we look for suitable* k_1 *and* k_2 *for*

$$z_{u_j} = u(x_j) - U_j,\ u(x) = \exp\,(-p_0x/\varepsilon)$$

$z_{w_j} = w(x_j) - W_j$, $w(x)$ is a function defined in Lemma 2 and U_j, W_j are approximate solutions of $u(x_j)$ and $w(x_j)$, respectively. In that way we prove Theorem 1 because of $|z_j| \leqslant M(|z_{u_j}| + |z_{w_j}|)$.

THEOREM 1. *Let* $\{v_j\}, j = 0(1)n+1$ *be the approximation to the solution* $y(x)$ *of* (1.1) *obtained using* (2.7). *Let* $p, f \in C^3\,[0, 1]$, $p(x) > \bar{p} > 0$ *and* $q \equiv 0$. *Then there are constants* σ *and* M *independent of* h *and* ε, *such that for* $j = 1(1)n$

$$\text{(3.5)}\quad \begin{cases} |v_j - y(x_j)| \leqslant Mh^2/(h+\varepsilon) + Mh^2\varepsilon^{-1}\exp(-\sigma x_j/\varepsilon) & \text{for } h \leqslant \varepsilon \\ |v_j - y(x_j)| \leqslant Mh + M\varepsilon\exp(-\sigma x_{j-1}/\varepsilon) & \text{for } \varepsilon \leqslant h. \end{cases}$$

Proof. We start with z_{w_j}. Let $h \leqslant \varepsilon$. Here, $\tau_j = \tau_j(w)$. In view of (3.2) and after several Taylor series expansions, at x_i, we have

$$T^2 = 3h\,[(\omega_{j-1} - 1)(A_{j-1} + (\omega_{j+1} + 1)/A_j]/2 + h/A_{j-1} - h/A_j -$$
$$- \varepsilon[1/(p_{j-1}A_{j-1}) + 1/(p_{j+1}A_j) + (2\omega_{j-1} - 1)/(p_jA_{j-1}\omega_j) +$$
$$+ (2\omega_{j+1} + 1)/(p_j\,A_j\,\omega_j)] = Q_0 + Q_1 + Q_2 + Q_3, \text{ where}$$
$$Q_0 = h(1/A_{j-1} - 1/A_j) + (3hp_{j-1}p_j\,p_{j+1}\,\omega_j - 2\,\varepsilon p_{j-1}\,p_{j+1})(A_{j-1} - A_j)/\bar{R}_j$$
$$\bar{R}_j = 2A_j \cdot A_{j-1}\,p_j\,p_{j-1}\,p_{j+1}\,\omega_j$$
$$Q_1 = 2p_{j-1}\,p_{j+1}\,(3\,\sigma_j - 2\,\varepsilon)/A_j(\omega_{j-1} - 2\omega_j + \omega_{j+1}) +$$
$$+ (A_{j-1} - A_j)(\omega_{j+1} - \omega_j)]/\bar{R}_j$$
$$Q_2 = \{3h^3\,\omega_j^2\,p_j^2\,(2A_j\,p''(\xi_3) - p'(\xi_1)\,A_j + 2A_{j-1}) +$$
$$+ 2\,\varepsilon h\omega_j\,p_j[(A_{j-1} - A_j)p'(\xi_2) + A_{j-1}(p'(\xi_1) - p'(\xi_2)) -$$
$$- 4hp''(\xi_3)A_j + 2hp'(\xi_1)p'(\xi_2)(A_j - A_{j-1})/p_j -$$
$$- 4A_{j-1}\,hp''(\xi_3)]\}/\bar{R}_j,$$

where

$$x_{j-1} \leqslant \xi_1 \leqslant x_j, x_j \leqslant \xi_2 \leqslant x_{j+1}, \xi_1 \leqslant \xi_3 \leqslant \xi_2$$
$$Q_3 = 6[2\omega_j\,A_j\,p_j\,(\sigma_j - \varepsilon) + (A_{j-1} - A_j)\,\omega_j\,p_j^2\,(\sigma_j - \varepsilon)]/R_j.$$

From (2.11), (2.12), (2.13), (2.16), (2.17), we have

$$|Q_0| \leqslant Mh^4\varepsilon^{-2}, \qquad |Q_1| \leqslant Mh^4\varepsilon^{-1}$$
$$|Q_2| \leqslant Mh^4\varepsilon^{-1}, \qquad |Q_3| \leqslant Mh^4\varepsilon^{-2}\,(h+\varepsilon)^{-1}$$

Then, from (2.9) we have

$$\text{(3.6)}\quad |T^2w^{(2)}(x_j)| \leqslant Mh^4\varepsilon^{-2}(h+\varepsilon)^{-1}\,(1 + \varepsilon^{-1}\exp(-\delta x_j/\varepsilon))$$
$$T^3 = h^2[A_j\,(\omega_{j+1} - \omega_{j-1}) + (A_{j-1} - A_j)\omega_j]/(2A_j\,A_{j-1})$$
$$+ \varepsilon h[(p_{j-1} - p_{j+1})\,A_{j-1} - p_{j+1}(A_j - A_{j-1})]/(p_{j+1}p_{j-1}A_jA_{j-1}).$$
$$\text{(3.7)}\quad |T^3w^{(3)}(x_j)| \leqslant Mh^4\varepsilon^{-2}(1 + \varepsilon^{-2}\exp(-\delta x_j/\varepsilon))$$

Now we consider the remainder terms in (3.2)

$$\text{(3.8)}\quad |r^-\,R_3(x_j,\,x_{j-1},\,w)| = |w^{\text{IV}}(\xi_j)\frac{h^4}{4!}r^-| \leqslant Mh^4\,\varepsilon^{-1}(1 + \varepsilon^{-3}\exp(-\delta\xi_j/\varepsilon)),$$

$x_{j-1} \leqslant \xi_j \leqslant x_j$, because of $|r^-| \leqslant M\varepsilon^{-1}$ and $\exp(\delta h/\varepsilon) \leqslant M$ (the others being similar).

From (3.6), (3.7) and (3.8), we obtain

$$\text{(3.9)}\quad |\tau_j(w)| \leqslant M\frac{h^2}{\varepsilon^2}\left(\frac{h^2}{\varepsilon+h} + \frac{h^2}{\varepsilon^2}\exp(-\sigma x_j/\varepsilon)\right),$$

and by Lemma 4 and Corollary 1, it follows that (3.5) holds for $|z_{w_j}|$.

Let $\varepsilon \leqslant h$. Then we have (from (3.4), (2.11), (2.14), (2.16), (2.17)).

$$|Q_0| \leqslant Mh^3\varepsilon^{-1}\exp(-\sigma hp_j/\varepsilon), \qquad |Q_1| \leqslant M\,\varepsilon^2\exp(-\sigma hp_j/\varepsilon)$$
$$|Q_2| \leqslant Mh^3, \qquad |Q_3| \leqslant Mh^2(\varepsilon+h)^{-1} \text{ and}$$
$$\text{(3.10)}\quad |T^2w''(x_j)| \leqslant Mh^2(\varepsilon+h)^{-1}(1 + \varepsilon^{-1}\exp(-\sigma x_j/\varepsilon))$$

As before, from the remainder terms we consider only

$$Y_j = -q^-\;p_{j-1}R_1(x_j,\,x_{j-1},\,w').$$
$$|Y_j| = \frac{1}{A_{j-1}}\int_{x_{j-1}}^{x_j}(x_j - s)\,|w^{(3)}(s)|ds \leqslant \frac{h}{A_{j-1}}\int_{x_{j-1}}^{x_j}(1 + \varepsilon^{-2}\exp(-\delta s/\varepsilon))ds$$
$$\leqslant M\,(h^2 + h^2\varepsilon^{-2}\exp(-\sigma x_{j-1}/\varepsilon)).$$

By partial integration for $j = 1$ we have

$$Y_j = -qp_{j-1}[hw''(x_j) + R_0(x_j,\,x_{j-1},\,w')] \text{ and}$$

$|Y_j| \leqslant M(h + \varepsilon h^{-1}\,\mu(\sigma)^{j-1})$. By Lemma 4 and Corollary 1, one finds that the contribution to the error from this term satisfies (3.5).

Estimate for z_{u_j}.

$$\text{(3.11)}\quad \begin{aligned} &\tau_j(u) = R_hu_j - Q_h(Lu_j),\; Lu_j = p_0\,\varepsilon^{-1}(p_0 - p_j)u_j,\; u_j = u(x_j) \\ &\tau_j(u) = \tau_r + \tau_q \end{aligned}$$

As in [2], we obtain that $\tau_r = \tau_1 + \tau_2$ (see (2.20), (2.18)), where

$$\tau_1 = u_j|r^-(\rho_j,\;\rho_j)\;\exp(p_0h/\varepsilon) - 1) + r^+(\rho_j,\;\rho_j)\;\exp(-p_0h/\varepsilon) - 1)]$$
$$\tau_2 = u_j|(\rho_{j-1} - \rho_j)D_r - (\rho_j,\rho_j)\exp(p_0h/\varepsilon) - 1) + (\rho_{j+1} - \rho_j)\,D_{r+}(\rho_j,\;\rho_j)\,\cdot$$
$$\cdot\exp(-p_0h/\varepsilon) - 1)] + N,$$

N denotes different expressions with the property that $|N| \leqslant Mh^4 \varepsilon^{-4} u_j$. D_{r-} and D_{r+} are determined by (2.19), (2.21).

$$\tau_q = u_j[-q^- p_0 \varepsilon^{-1} (p_0 - p_{j-1}) \exp(hp_0/\varepsilon) - q^c p_0 \varepsilon^{-1}(p_0 - p_j) -$$
$$- q^+ p_0 \varepsilon^{-1} (p_0 - p_{j+1}) \exp(-p_0 h/\varepsilon)]$$

Let $h \leqslant \varepsilon$. After expanding in Taylor series according to (2.23), we have

$$\tau_j = u_j \exp(-p_0 h/\varepsilon) r^+(\rho_j, \rho_j) \exp(p_0 h/\varepsilon) - 1) [\exp(-h(p_j - p_0)/\varepsilon) - 1] =$$
$$= u_j(p_0 h/\varepsilon - p_0^2 h^2 \varepsilon^{-2}/2)(-(p_j - p_0)h/\varepsilon + (p_j - p_0)^2 h^2 \varepsilon^{-2}/2) \cdot$$
$$\cdot (\rho_j/h + \rho_j^2/h - \rho_j^4/(3h)) + N$$

(3.12) $$\tau_1 = -u_j p_0 h \varepsilon^{-2} \rho_j(p_j - p_0) + N$$

(3.13) $$\tau_2 = 3u_j p_0 \varepsilon^{-1}[(\omega_j^2 - 1)\omega_j(-\rho_{j-1} + 2\rho_j - \rho_{j+1}) + (\omega_j^2 - 1)(\rho_{j+1} - \rho_{j-1})]/(3\omega_j^2 - 1)^2 + N$$

(3.14) $$|\tau_2| \leqslant Mh^4 \varepsilon^{-4} u_j$$

Further,

$$-\tau_q = u_j \frac{p_0}{\varepsilon}(p_0 - p_j)|q^- + q^c + q^+ + p_0 \frac{h}{\varepsilon}(q^- - q^+) + M(q^- + q^+)h^2 \varepsilon^{-2}|.$$

By applying (2.25)—(2.29) and after some Taylor developments, we obtain

$q^- + q^c + q^+ = 2\rho_j^2/p_j + N_1$, $|q^- - q^+| \leqslant Mh^3 \varepsilon^{-2}$, $|N_1| \leqslant Mh^4 \varepsilon^{-4}$

$q^+ + q^- \leqslant Mh^2$ and thus

(3.15) $$\tau_q = u_j \rho_j p_0 h \varepsilon^{-2}(p_j - p_0) + N$$

According to (3.11), (3.12), (3.14), (3.15), Lemma 4 leads to a term in the error estimate which satisfies (3.5).

Let $\varepsilon \leqslant h$. Then

$$r^-/r^+ = \exp(-hp_0/\varepsilon) + hx_j p'(\theta_0)\varepsilon^{-1} \exp(-\theta_1 h p_j \varepsilon^{-1}) + Mh^2 \varepsilon^{-1},$$

$0 < \theta_i < 1$ $(i = 0, 1, 2)$

$$\tau_r = r^+ u_j \left[\frac{r^-}{r^+}(\exp(p_0 h/\varepsilon) - 1) + \exp(p_0 h/\varepsilon) - 1\right]$$

$$|\tau_r| \leqslant M\varepsilon r^+ \exp(-p_0 x_{j-1}/\varepsilon) \exp(p_0 h(\theta_2 - 1)/\varepsilon).$$

$$(h^2 x_j \varepsilon^{-2} \exp(-h\delta/\varepsilon) + h^3/\varepsilon^2), \quad \delta > 0$$

(3.16) $$|\tau_r| \leqslant M\varepsilon h^{-1} \exp(-\sigma x_{j-1}/\varepsilon)$$

(3.17) $$|\tau_q| \leqslant M\exp(-p_0 x_{j-1}/\varepsilon)|x_{j-1}p'(\xi_1) + x_j p'(\xi_2)\exp(-p_0 h/\varepsilon) + x_{j+1}p'(\xi_3)\exp(-2p_0 h/\varepsilon)| \leqslant M\varepsilon h^{-1}\exp(-\sigma x_{j-1}/\varepsilon)$$

As before, from (3.16) and (3.17) one finds that the contribution to the error from these terms satisfies (3.5).

The proof is complete.

The estimate (3.5) is the same as the one which was obtained by Kellogg and Tsan ([12]) for Allen-Southwel — Il'in's scheme.

4. Accuracy of the approximate solution between the grid points. The approximate solution between the grid points is defined by (1.2) where constants $v_j^{(1)}$, $j = 0(1)n$ can be determined from the relations:

$$v_{j-1}^{(1)} = (v_j - v_{j-1} - s_j)/\gamma_j, \qquad j = 1(1)n,$$

(4.1) $$a_j v_j^{(1)} = b_j v_{j-1}^{(1)} + r_j, \qquad j = 1(1)n$$

$$s_j = \frac{h^2}{6}\left(2\frac{f_{j-1}}{\sigma_{j-1}} - \frac{p_j r_j}{\sigma_j a_j} + \frac{f_j}{\sigma_j}\right), \quad r_j = \frac{h}{2}\left(\frac{f_{j-1}}{\sigma_{j-1}} + \frac{f_j}{\sigma_j}\right)$$

After that, we obtain $v_j^{(2)}(j = 0(1)n)$ from (2.1), $v_j^{(3)}(j = 1(1)n - 1)$ from (2.5) and $v_n^{(3)}$ from (2.2).

Denote $z^{(k)}(x) = y^{(k)}(x) - v^{(k)}(x)$. Then for $x \in [x_j, x_{j+1}]$, we have

(4.2) $$z(x) = z_j + (x - x_j)z_j^{(1)} + \frac{(x - x_j)^2}{2} z_j^{(2)} + \frac{(x - x_j)^3}{3!} z_j^{(3)} + R_3(x_j, x, y)$$

From the construction of the scheme [10], [15], the following holds:

(4.3) $$z_{j-1}^{(1)} = (z_j^{(0)} - z_{j-1}^{(0)} - \varphi_{2,j})/\gamma_j, \quad j = 1(1)n$$

(4.4) $$a_j z_j^{(1)} = b_j z_{j-1}^{(1)} + \varphi_{1,j}, \qquad j = 1(1)n$$

(4.5) $$\varphi_{2,j} = \psi_{0,j} + h^2[\eta_{j-1}/(3\sigma_{j-1}) + \eta_j/(6\sigma_j) - \psi_{2,j}/6 - p_j \varphi_{1,j}/(6a_j\sigma_j)]$$

(4.6) $$\varphi_{1,j} = \psi_{1,j} - h\psi_{2,j}/2 + h(\eta_{j-1}/\sigma_{j-1} + \eta_j/\sigma_j)/2$$

(4.7) $$\eta_j = y_j''(\sigma_j - \varepsilon)$$

(4.8) $$\psi_{i,j} = R_{3-i}(x_{j-1}, x_j, y^{(i)})$$

(4.9) $$\sigma_j z_j^{(2)} = \eta_j - p_j z_j^{(1)}, \qquad j = 0(1)n$$

(4.10) $$z_{j-1}^{(3)} = z^{(3)}(x_{j-1} + 0) = (z_j^{(2)} - z_{j-1}^{(2)} - \psi_{2,j})h, \quad j = 1(1)n.$$

Since

$$R_{3-i}(x_j, x, y^{(i)}) = R_{3-i}(x_j, x, u^{(i)}) + R_{3-i}(x_j, x, w^{(i)}) \text{ and}$$

$$R_{3-i}(x_j, x, u^{(i)}) = \frac{\exp(-p_0 t)}{(3 - i)!} \sum_{k=0}^{3-i} \left(\frac{p_0}{\varepsilon}\right)^{3-k} \frac{(3 - i)!}{(3 - i - k)!}(x - x_j)^{3-i-k} -$$

$$-\left(\frac{p_0}{\varepsilon}\right)^i \exp(-p_0 t_j) = (x-x_j)^{4-i}\varepsilon^{-4}\cdot\exp(-p_0\xi/\varepsilon),\quad x_j < \xi < x,$$

$$t = x/\varepsilon,\quad t_j = x_j/\varepsilon,$$

by using the first equality when $\varepsilon \leqslant h$ and the second when $h \leqslant \varepsilon$, we have for $j = 0(1)\,n$ ((4.3)–(4.8), (2.11), (3.5)):

$$|\psi_{i,j}| \leqslant \begin{cases} M[\varepsilon^{-i}\exp(-\sigma t_{j-1}) + h^{4-i}], & \varepsilon \leqslant h,\\ M[h^{4-i}\varepsilon^{-4}\exp(-\sigma t_j) + h^{4-i}], & h \leqslant \varepsilon\end{cases}$$

$$|z_j^{(1)}| \leqslant \begin{cases} M[h(\varepsilon+h)^{-1} + h^3\varepsilon^{-4}\exp(-\sigma t_{j+1})], & h \leqslant \varepsilon,\\ M[1 + h\varepsilon^{-2}\exp(-\sigma t_j)], & \varepsilon \leqslant h,\end{cases}$$

$$|z_j^{(2)}| \leqslant \begin{cases} M[h\varepsilon^{-1}(\varepsilon+h)^{-1} + h^3\varepsilon^{-5}\exp(-\sigma t_{j+1})], & h \leqslant \varepsilon\\ M[(\varepsilon+h)^{-1} + \varepsilon^{-2}\exp(-\sigma t_j)], & \varepsilon \leqslant h,\end{cases}$$

$$|z_j^{(3)}| \leqslant \begin{cases} M[\varepsilon^{-1}(\varepsilon+h)^{-1} + h^2\varepsilon^{-5}\exp(-\sigma t_{j+1})], & h \leqslant \varepsilon,\\ M[h^{-1}(\varepsilon+h)^{-1} + h^{-1}\exp(-\sigma t_j)], & \varepsilon \leqslant h,\end{cases}$$

σ is a constant independent of ε and h. Thus, from (4.2) we obtain that the following theorem holds.

THEOREM 2. *Let p, $f \in C^3[0, 1]$, $p(x) \geqslant \bar{p} > 0$ and $q \equiv 0$. Then there are constants σ and M independent of h and ε, such that for $x \in [x_j, x_{j+1}]$, $j = 0(1)n$, the estimate*

$$|y(x) - v(x)| \leqslant \begin{cases} M[h^2(\varepsilon+h)^{-1} + (x-x_j)h^3\varepsilon^{-4}\exp(-\sigma x_{j+1}/\varepsilon)], & h \leq \varepsilon\\ M[h + (x-x_j)\,h\varepsilon^{-2}\exp(-\sigma x_j/\varepsilon)] & \varepsilon \leq h\end{cases}$$

(4.11)

is valid.

Thus, for $x = x_j$ we have a uniform convergence. The same holds for $x \in [\alpha, 1]$, where α is a constant independent of ε and h. The uniform convergence in boundary layer is not achieved.

THEOREM 3. *Let $p(x) = p = \text{const}$, $p > 0$, $q(x) \equiv 0$, $f \equiv 0$, $\delta_0 = 1$, $\delta_1 = \exp(-1/\varepsilon)$. Then spline $v(x)$ is an interpolation spline for solution $y(x) = \exp(-px/\varepsilon)$ with properties:*

$$(4.12)\qquad |y(x) - v(x)| \leqslant \begin{cases} M(x-x_j)h^3\varepsilon^{-4}\exp(-p\theta/\varepsilon), & h \leqslant \varepsilon\\ M(x-x_j)h\varepsilon^{-2}\exp(-p\theta/\varepsilon), & \varepsilon \leqslant h,\end{cases}$$

$x \in [x_j, x_{j+1}]$, $x_j \leqslant \theta \leqslant x_{j+1}$.

$$(4.13)\qquad y'(x_j) - v_j^{(1)} = \exp(-ph/\varepsilon)(y'(x_{j-1}) - v_{j-1}^{(1)}),\ j = 0(1)n.$$

Proof. **In this case $\tau_q = \tau_r = 0$ (see 3.11). The matrix of the system $R_h(y(x_j) - v(x_j)) = \tau_j$, $j = 1(1)n$ is inverse monotone, and we have $v(x_j) = y(x_j)$, $j = 0(1)n + 1$. The estimate (4.12) is obtained from (4.2)–(4.10). Since, $r_j = 0$ in (4.1) and $b_j/a_j = \exp(-ph,\ \varepsilon)$ we have (4.13), i.e. $\varphi_{i,j} = 0$ in (4.4).**

THEOREM 4. *There are no constants M and σ independent of ε and h, such that*

$$|A_j T_j^2| \leqslant M h^{1+\sigma}$$

Proof. Suppose the contrary, and consider the case from Theorem 3. Then, $q^c = 2(q^+ + q^-) = \dfrac{4}{pA} > 0$, $A = A_j = \text{const}$. From the expressions for T_j^1 and T_j^2 in section 3, we have

$$(4.14)\qquad 2r^- = (-ph^{-1} + 2\varepsilon h^{-2})(q^- + q^c + q^+) - h^{-1}T_j^1 + 2h^{-2}T_j^2$$

Multiplying (4.14) by pA we obtain

$$2r^- pA = 6\cdot p^{-1}(-p + 2c) + 0(h) + 0\,(h)^{-1+\sigma},\quad \varepsilon = ch.$$

For $c < p/2$ and h sufficiently small we have that $r^- < 0$, which is a contradiction. This suggests that $0(h)$ is the best uniform order of accuracy for our scheme.

5. Numerical examples. **In the examples we shall give, the order of uniform convergence according to [5] was determined experimentally. We used the equidistant mesh with $h = 1/2^{k+3}$, $k = 0(1)6$. By $p_{k,\varepsilon}$ we denote the estimates of order of classical convergence which are obtained** for different k and ε:

$$p_{k,\varepsilon} = \log_2(z_{k,\varepsilon}/z_{k+1,\varepsilon}),\ \text{where}\ z_{k,\varepsilon} = \max_j |v_{j,k} - v_{2j,k+1}|,$$

and $v_{j,k}$ is the approximate solution obtained with the step $h = 1/2^{k+3}$. $\bar{p}_\varepsilon$ denotes the mean value of the estimate of the order of classical convergence for a fixed ε. For an estimate of the order of classical convergence $\bar{p}_\varepsilon$ is taken for the largest ε, **and for the estimate of the order of uniform** convergence $\bar{p}_\varepsilon$ is taken for the smallest ε. Tables 1 and 2 contain $p_{k,\varepsilon}$.

Problem 1 ([5]):

$$\varepsilon y'' + (1+x^2)y' - (x-1/2)^2 y = -4(3x^2 - 3x + 1)((x-1/2)^2 + 2),$$

$$y(0) = -1,\ y(1) = 0$$

Table 1

ε \ k	0	1	2	3	4	$\bar{p}_\varepsilon$
1/2	2.02	2.00	2.00	2.04	1.88	1.99
1/4	2.01	2.02	2.00	2.00	1.98	2.00
1/8	2.22	2.06	2.02	2.00	2.00	2.06
1/16	2.25	1.88	1.97	1.99	2.01	2.02
1/32	2.02	1.79	1.91	1.98	1.99	1.94
1/64	1.45	1.86	1.74	1.92	1.98	1.79
1/128	1.08	1.46	1.72	1.76	1.93	1.59
1/256	1.02	1.08	1.47	1.65	1.77	1.40
1/512	1.02	1.02	1.07	1.48	1.62	1.24

The computed estimate of the order of uniform convergence is 1.24, and the classical one 1.99.

Problem 2. ([8]) : $\varepsilon y'' + y' - (1 + \varepsilon)y = 0$

$$y(0) = 1 + \exp(-1),\ y(1) = \exp(-(1 + \varepsilon)/\varepsilon) + 1$$

Exact solution :

$$y(x) = \exp(-x(1 + \varepsilon)/\varepsilon) + \exp(x - 1)$$

Table 2

ε \ k	0	1	2	3	4	$\bar{p}_\varepsilon$
1/2	2.02	1.99	1.93	3.92	−1.52	1.98
1/4	1.98	2.02	1.99	1.92	3.44	1.98
1/8	2.20	2.98	2.02	1.99	1.82	1.99
1/16	1.75	2.18	1.99	2.02	1.96	1.98
1/32	1.21	1.79	2.17	2.00	2.04	1.84
1/64	1.09	1.22	1.77	2.17	2.00	1.65
1/128	0.98	1.18	1.23	1.78	2.19	1.47
1/256	0.88	1.01	1.21	1.24	1.78	1.22
1/512	0.88	0.95	1.03	1.23	1.24	1.07

The computed order of uniform convergence is 1.07, and the classical one 1.98. Estimates for $p_{k,\varepsilon}$ departing significantly from the mean value have not been taken into account in calculating $\bar{p}_\varepsilon$. Table 3 contains some values $|\varepsilon z'(x_j)| = \varepsilon|y'(x_j) - v_j^{(1)}|$ for different h.

Table 3

$\varepsilon = 1/512$

x_j \ h	1/32	1/128	1/512
0	0.8115087E+00	0.3004345E+00	0.4691537E−02
1/16	0.5628536E−05	0.1010988E−06	0.7615551E−08
2/16	0.1420366E−05	0.2135444E−06	0.1621974E−07
3/16	0.2228509E−05	0.3400590E−06	0.2590200E−07
4/16	0.3134497E−05	0.4819887E−06	0.3676562E−07
5/16	0.4147606E−05	0.6407947E−06	0.4892268E−07
6/16	0.5277216E−05	0.8180623E−06	0.6249483E−07
7/16	0.6534747E−05	0.1015511E−05	0.7761404E−07
8/16	0.7931727E−05	0.1235006E−05	0.9442339E−07
9/16	0.9480859E−05	0.1478568E−05	0.1130780E−06
10/16	0.1119591E−04	0.1748387E−05	0.1337460E−06
11/16	0.1309176E−04	0.2046835E−05	0.1566093E−06
12/16	0.1518456E−04	0.2376482E−05	0.1818654E−06
13/16	0.1749175E−04	0.2740108E−05	0.2097276E−06
14/16	0.2003222E−04	0.3140724E−05	0.2404271E−06
15/16	0.2282637E−04	0.3581586E−05	0.2742139E−06
1	0.2589657E−04	0.4066366E−05	0.3113064E−06

REFERENCES

[1] B e r g e r, A. E., S o l o m o n, J. M., L e v e n t a l, S. H. & W e i n b e r g, B. C., *Generalized operator compact implicit schemes of boundary layer problems*, Math. Comput. **35**, (1980), 695—731.

[2] B e r g e r, A. E., S o l o m o n, J. M. & C i m e n t, M., *An Analysis of a Uniformly Accurate Difference Method for a Singular Perturbation Problem*, Math. Comput., **37**, (1981), 79—94.

[3] B o g l a e v, D. J., *Variacionno-raznostnaya shema dlya kraevoĭ zadachi s malym parametrom pri starsheĭ proizvodnoĭ*, Zh. Vychisl. Mat. i. Mat. Fiz. **21** (1981), 887—896.

[4] C h i n, Y. C. R. & K r a s n y, R., *A hybrid asymptotic finite element method for stiff two-point boundary value problems*. SIAM J. Sci. Stat. Comput., **4** (1983), 229—243.

[5] D o o l a n, E. P., M i l l e r, J. J. H., S c h i l d e r s, W. H. A., *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Dublin, Boole Press 1980.

[6] E m e l ' y a n o v, K. V., *O raznostnoĭ sheme dlya obyknovenogo differencial 'nogo uravneniya s malym parametrom*, Zh. Vychisl. Mat. i Mat. Fiz. **18** (1978), 1146—1153.

[7] F l a h e r t y, J. E. & M a t h o n, W., *Collocation with polynomial and tension splines for singularly-perturbed boundary value problems*, SIAM J. Sci. Stat. Comput, 260—289.

[8] D e G r o e n, P. P. N., & H e m k e r, P. W., *Error bounds for exponentially fitted Galerkin methods applied to stiff two-point boundary value problems*, Proc. Conf. on Numerical Analysis of Singular Perturbation Problems, May 20—June 2, 1978, University of Nijmegen The Netherlands (P. W. Hemker and J. J. Miller, eds.), North-Holland, Amsterdam, 215—249 (1979).

[9] I l ' i n, A. M., *Raznostnaya shema dlja differencialnogo uravneniya s malym parametrom pri starsheĭ proizvodnoĭ*, Mat. zametki., **6** (1969), 237—248.

[10] I l ' i n, U. P., O *splainovyh resheniyah obyknovennyh differencial'nyh uravnenii*. Zh. Vychisl. Mat. i. Mat. Fiz., (1978), **18** 620—627.

[11] J a i n, M. K., & A z i z, T., *Numerical solution of stiff and convection diffusion equations using adaptive spline function approximation.*, Apl. Math. Modelling, **7** (1983), 57—62.

[12] K e l l o g, R. B. & T s a n, A., *Analysis of some difference approximations for a singular perturbation problem without turning points*, Math. Comput., **32** (1978), 1025—1039.

[13] L o r e n z, J., *Stability and Consistency Analysis of Difference Methods for Singular Perturbation Problems* Proc. Conf. on Analytical and Numerical Approaches to Asymptotic Problems in Analysis, June 9—13, 1980, University of Nijmegen, The Netherlands (O. Axelsson, L. Frank and A. Van der Sluis, Esd.), North-Holland, Amsterdam, 141—155 (1981).

[14] R i o r d a n, E. O., *Singularly Perturbed Finite Element Methods*, Numer. Math. **44** (1984), 425—434.

[15] S u r l a, K., *Accuracy increase for some spline solutions of two-point boundary value problems*, Zb. rad. Prir.-Mat. Fak. Univ. u Novom Sadu, Ser. mat. **14** (1) (1984), 51—61.

[16] S u r l a, K., *A uniformly convergent spline difference scheme for singular perturbation problems*, ZAMM. **66** (1986), T. 328—T 329.

[17] Z a v y a l o v, Y u. S., K v a s o v, B. I. & M i r o s h n i c h e n k o, Z. L., *Metody splaĭn funkcii*, Moskva, 1980.

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Institute of Mathematics
Dr. Ilije Djuričića 4
YU — 21000 Novi Sad
Yugoslavia