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SOME GENERALIZATIONS OF JESSEN'S INEQUALITY

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1. The inequality of Jessen is a generalization of that of Jessen (see [1]). In what follows we want to extend this inequality by replacing the isotony required in Jessen's inequality with a weaker condition. This allows the passage to inequalities for convex functions of higher orders.

2. Let us recall some notations and definitions. We consider the set $C = C[a, b]$ of all continuous real functions defined on $[a, b]$ and the set K of convex functions (from C).

Let also $e_k(k = 0, 1, \ldots)$ and w_c (with $c \in (a, b)$) be the functions defined by :

$$e_k(x) = x^k, \quad \forall x \in [a, b]$$

respectively

$$w_c(x) = |x - c|, \quad \forall x \in [a, b]$$

A functional $A : C \to R$ is linear if :

$$A(af + bg) = aA(f) + bA(g), \quad \forall f, g \in C\,; \ a, b \in R$$

and it is isotonic if :

$$A(f) \geqslant 0, \quad \forall f \geqslant 0.$$

We consider the following form of Jessen's inequality :

THEOREM 1. *The function $f \in C$ is convex if and only if for any isotonic linear functional A, with $A(e_0) = 1$, f verifies :*

$$f(A(e_1)) \leqslant A(f) \tag{1}$$

Remark 1. As w_c is convex for any c, we have also :

$$w_c(A(e_1)) \leqslant A(w_c) \tag{2}$$

We want to prove that (2) can replace the condition of isotony of A in (1). For this we need the following theorem of K. Toda [6] and T. Popoviciu [4] :

THEOREM 2. *Every function $f \in K$ is the uniform limit of a sequence $(g_m)_{m\geq 1}$, given by* :

$$g_m = p_m \cdot e_0 + q_m \cdot e_1 + \sum_{k=0}^{m} p_{k,m} \cdot w_{c_{k,m}} \tag{3}$$

where $p_m, q_m \in R$, $p_{k,m} \geq 0$, $c_{k,m} \in [a, b]$.

Using this theorem, in [7] it is proved the following result.

THEOREM 3. *Let A be a linear and continuous operator defined on C. Then,*

$$A(f) \geq 0, \quad \forall f \in K$$

if and only if :

$$A(e_0) = A(e_1) = 0, \quad A(w_c) \geq 0, \quad \forall c \in [a, b].$$

Similarly we can prove the following generalization of Theorem 1. We define by L^+ the set of liniar and continuous functionals A, which satisfy $A(e_0) = 1$ and the relation (2).

THEOREM 4. *The function $f \in C$ is convex if and only if for any $A \in L^+$, f verifies* (1).

In fact we can prove a stronger result. Let S^+ denote the set of all superadditive, positively homogeneous, upper semicontinuous functionals A, which satisfy (2) and $A(ae_0 + be_1) \geq a + b \cdot A(e_1)$.

THEOREM 5. *The function $f \in C$ is convex if and only if for any $A \in S^+$, f verifies* (1).

Proof. The sufficiency is obviously : take $A(f) = sf(x) + (1-s)f(y)$ with $s \in (0, 1)$, $x, y \in [a, b]$.

The necessity : for a given convex function f, let the sequence $(g_m)_{m\geq 1}$ given by (3), which converges uniformly to f. If $A \in S^+$, we have :

$$A(g_m) \geq p_m + q_m \cdot A(e_1) + \sum_{k=0}^{m} p_{k,m} \cdot A(w_{c_{k,m}}) \geq g_m(A(e_1)).$$

As A is upper semicontinuous it follows :

$$A(f) \geq \lim_{m\to\infty} A(g_m) \geq \lim_{m\to\infty} g_m(A(e_1)) = f(A(e_1))$$

We remark that the converse inequality of (1) may be also used for the characterization of the convexity. So, let S^- denote the set of all subadditive, positively homogeneous, lower semicontinuous functionals A, which satisfy $A(a \cdot e_0 + b \cdot e_1) \leq a + b \cdot A(e_1)$ and :

$$w_c(A(e_1)) \geq A(w_c) \tag{2'}$$

THEOREM 6. *The function $f \in C$ is convex if and only if for any $A \in S^-$, f verifies* :

$$f(A(e_1)) \geq A(f). \tag{1'}$$

3. As we have proved in [5], the convexity of order two may be characterized by the same relation (1) valid for some linear functionals which verify the conditions

$$A(e_0) = 1, \; A(e_2) = [A(e_1)]^2$$

and, of course, are not isotonic. In what follows we want to transpose theorem 5 to convexity of higher order. We need the following result from [2] which generalizes Theorem 2.

Let us denote by w_c^n the function defined by :

$$w_c^n(x) = \begin{cases} 0 & \text{if} \quad x < c \\ (x-c)^{n-1} & \text{if} \quad x \geq c \end{cases}$$

by P_n, the set of polynomials of degree at most n and by $K_n = K_n[a, b]$ the set of all n-convex functions (convex of order n).

THEOREM 7. *Every function from $K_n (n \geq 1)$ can be* ***approximated*** *uniformly on $[a, b]$ by spline functions of the form* :

$$g_{m,1}(x) = p_{m,n}(x) + \sum_{k=1}^{1-n} q_{m,1,n,k} \cdot w_{c_k}^n(x) \tag{3'}$$

where $p_{m,n} \in P_{n-1}$ *and* $q_{m,1,n,k} > 0$.

Using this result, we obtain a direct generalization of Theorem 4 in :

THEOREM 8. *The function $f \in C$ is in K_n if and only if for any continuous linear functional $A : C \to R$ with the properties* :

$$A(p) \geq p(A(e_1)), \quad \forall p \in P_{n-1} \tag{4}$$

and

$$w_c^n(A(e_1)) \leq A(w_c^n), \quad \forall c \in (a, b) \tag{5}$$

the function f verifies :

$$f(A(e_1)) \leq A(f)$$

In fact, we can prove the following general result which extends also Theorem 5 : let S_n^+ denote the set of all superadditive, positively homogeneous, upper semicontinuous functionals, $A : C \to R$, which satisfy (4) and (5).

THEOREM 9. *The function $f \in C$ is in K_n if and only if for any $A \in S_n^+$, it verifies* (1).

Inequality (1') may be also used : let S_n^- denote the set of subadditive, positively homogeneous, lower semicontinuous functionals $A : C \to R$ which satisfy :

$$A(p) \leq p(A(e_1)) \tag{4'}$$

and

$$w_c^n(A(e_1)) \geq A(w_c^n), \quad \forall c \in (a, b). \tag{5'}$$

THEOREM 10. *The function $f \in C$ is in K_n if and only if for any $A \in S_n^-$ it verifies* (1').

In the same manner, we can give the following generalization of the main result from [2], which extends also Theorem 3.

THEOREM 11. *Let $B : C \to R$ be a superadditive, positively homogeneous, upper semicontinuous functional. In order that $B(f) \geq 0$ for every $f \in K_n$ $(n \geq 1)$ it is necessary and sufficient that:*

$$B(p) \geq 0, \quad \forall p \in P_{n-1} \tag{6}$$

and

$$B(w_c^n) \geq 0, \quad \forall c \in (a, b). \tag{7}$$

Remark 2. There is a strong connection between the functionals A from Theorem 9 and the functionals B from Theorem 10.

If A satisfies (4) and (5), then

$$B(f) = A(f) - f(A(e_1))$$

verifies (6) and (7). Conversely, if B has properties (6) and (7) and $B(e_1) = 0$, then

$$A(f) = B(f) + f(B(e_1))$$

verifies (4) and (5).

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