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DUALITY RELATIONS AND CHARACTERIZATIONS OF BEST APPROXIMATION FOR p-CONVEX SETS

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1. Introduction. A subset of a vector space is said to be *convex* if together with any two of its points it contains the whole line segment joining them. J. von Neumann [16], in connection with the introduction of locally convex topologies, required that only the midpoint of this segment belongs to the given set, defining so the $\frac{1}{2}$-convex sets. J. W. Green and W. Gustin [8] defined and studied the quasiconvex sets: a set in a vector space is called *quasiconvex* if together with any two of its points x and y it contains all the points dividing the segment $[x, y]$ into a ratio belonging to a prescribed set $\Delta \subset]0, 1[$. Some extensions of the notion of quasiconvexity were given by A. Aleman [1] and Gh. Toader [22].

This paper is concerned with p-convex sets, i.e., quasi-convex sets with $\Delta = \{p\}$, where p is a given number in $]0,1[$. In other words, a set Y in a vector space is said to be *p-convex* if $pY + (1 - p)\,Y \subset Y$. The $\frac{1}{2}$-convex sets or midconvex or also centred-convex [17]) were recently used in the study of the continuity of Jensen convex functions ([5], [21]) and of the stability of Jensen inequality ([3]). One of the authors of this paper proved in [14] some separation and support theorems for p-convex sets in topological vector spaces, and gave in [15] a Lagrange multiplier rule in p-convex programming.

N. P. Korneichuk [12], pp. 28—33, proved the following "duality relations" for the best approximation by elements of convex sets:

THEOREM 1.1. *If Y is a nonvoid convex subset of a real normed space X, then*

a) *For every $x \in X$, the duality relation*

$$\inf\{\|x - y\| : y \in Y\} = \sup\{x^*(x) - \sup\{x^*(y) : y \in Y\} : x^* \in B^*\},$$

holds, where $B^* = \{x^* \in X^* : \|x^*\| \leqslant 1\}$ is the closed unit ball in the dual space X^* of X; and

b) *For every* $x \in X \setminus Y$, *there exists and* x_0^* *in* X^* *with* $\|x_0^*\| = 1$ *such that*

$$\inf\{\|x - y\| : y \in Y\} = x_0^*(x) - \sup\{x_0^*(y) : y \in Y\}.$$

In Section 2 of this paper we shall show that Theorem 1.1 remains true if the set Y is supposed to be only p-convex. Section 3 contains extensions to the p-convex case of the results of N. P. Korneĭchuk, A. A. Ligun and V. G. Doronin [13] concerning the best approximation with conical constraints. Section 4 deals with best approximations by elements of p-caverns (subsets of a normed space whose complement is a bounded p-convex set with nonvoid interior) extending some results proved by C. Franchetti and I. Singer [6] in the convex case.

We shall need the following three known results:

THEOREM 1.2 ([14], Theorem 4.4). *If* Y *is a nonvoid* p*-convex subset of a real locally convex space* X *and* $x_0 \in X \setminus \bar{Y}$, *then there exists* $x^* \in X^*$ *such that* $\inf\{x^*(y) : y \in Y\} > x^*(x_0)$.

THEOREM 1.3 ([14], Theorem 1.2). *If* Y *is a* p*-convex subset with nonvoid interior of a real locally convex space, then every boudary point of* Y *is contained in a closed hyperplane supporting* Y.

THEOREM 1.4 ([1], Theorem 3.3). *If* Y *is a* p*-convex subset of a topological vector space, then*

a) *the closure* $\bar{Y}$ *of* Y *is convex;*

b) *if* $x \in \bar{Y}$, $y \in \operatorname{int} Y$ *and* $0 < \alpha < 1$, *then* $\alpha x + (1-\alpha) y \in \operatorname{int} Y$; *in particular, the interior of* Y *is convex.*

All the vector spaces considered in this paper will be taken over the field R of real numbers.

2. Duality relations for p-convex sets. We need the following extension of a well-known separation theorem for p-convex sets:

THEOREM 2.1. *Let* Y *and* Z *be two nonvoid and disjoint subsets of a topological vector space* X. *If* Y *is* p*-convex and* Z *is convex and open, then* Y *and* Z *can be separated by a closed hyperplane in* X.

Proof. By Theorem 1.4, *a*), the closure $\bar{Y}$ of Y is convex. We state that $\bar{Y} \cap Z = \emptyset$. Indeed, if $x \in \bar{Y} \cap Z$, then, since Z is a neighbourhood of x, it would follow that $Z \cap Y \neq \emptyset$, which contradicts the hypothesis of the theorem. Now, applying a classical separation theorem ([9], Theorem 9.1), the sets $\bar{Y}$ and Z can be separated by a closed hyperplane in X. It follows that Y and Z are separated too by this hyperplane. ∎

The next lemma is well known and easy to prove (see [12], p. 30):

LEMMA 2.2. *If* X *is a normed space,* $r > 0$ *and* $x^* \in X^*$, *then*

$$\sup\{x^*(x) : x \in X,\ \|x\| < r\} = r \cdot \|x^*\|.$$

Now, we state the first duality theorem:

THEOREM 2.3. *If* Y *is a nonvoid* p*-convex subset of a normed space* X *and* $x \in X$, *then the following duality relation holds:*

(1) $$\inf\{\|x - y\| : y \in Y\} = \sup\{x^*(x) - \sup\{x^*(y) : y \in Y\} : x^* \in B^*\},$$

where $B^* = \{x^* \in X^* : \|x^*\| \leqslant 1\}$ *is the closed unit ball in the dual space* X^* *of* X.

Proof. Put

$$i = \inf\{\|x - y\| : y \in Y\},\quad s = \sup\{x^*(x) - \sup\{x^*(y) : y \in Y\} : x^* \in B^*\}.$$

Since $x^*(x) - \sup\{x^*(y) : y \in Y\} \leqslant x^*(x) - x^*(y') \leqslant \|x - y'\|$ for all $x^* \in B^*$ and all $y' \in Y$, we have $s \leqslant i$. As $0 \in B^*$, it follows that $s \geqslant 0$, hence $i = s$ in the case $i = 0$. Suppose now that $i > 0$. By Theorem 2.1, the p-convex set Y and the open ball

$$Z = \{z \in X : \|x - z\| < i\}$$

can be separated by a closed hyperplane. Therefore, there exist $x_0^* \in X^*$ with $\|x^*\| = 1$ and $c \in R$ such that

(2) $$x_0^*(y) \leqslant c < x_0^*(z) \text{ for all } y \in Y \text{ and all } z \in Z.$$

By (2) and Lemma 2.2, we obtain

$$\sup\{x_0^*(y) : y \in Y\} \leqslant \inf\{x_0^*(z) : z \in Z\} = \inf\{x_0^*(x - w) : w \in X,$$
$$\|w\| < i\} = x_0^*(x) - \sup\{x_0^*(w) : w \in X,\ \|w\| < i\} = x_0^*(x) - i \cdot \|x_0^*\|,$$

which implies that

$$i = i \cdot \|x_0^*\| \leqslant x_0^*(x) - \sup\{x_0^*(y) : y \in Y\} \leqslant s.$$

The inequalities $s \leqslant i$ and $i \leqslant s$ give $i = s$ and Theorem 2.3 is proven. ∎

The following theorem contains a condition ensuring that the supremum in the right-hand of relation (1) is attained:

THEOREM 2.4. *Adding to the hypotheses of Theorem 2.3 the condition* $x \notin \bar{Y}$, *there exists* $x_0^* \in X^*$ *with* $\|x_0^*\| = 1$ *such that the second duality relation holds:*

(3) $$\inf\{\|x - y\| : y \in Y\} = x_0^*(x) - \sup\{x_0^*(y) : y \in Y\}.$$

Proof. Keeping the notation in the proof of Theorem 2.3, there exists a sequence $(x_n^*)_{n \in N}$ in B^* such that

(4) $$\lim_{n \to \infty}[x_n^*(x) - \sup\{x_n^*(y) : y \in Y\}] = s.$$

By the Alaoglu-Bourbaki theorem, the closed unit ball B^* is w^*-compact, so that there exist a subnet $(x^*_{n_k})_{k \in K}$ (K is a directed set) of the sequence (x_n^*) and an element x_0^* of B^* such that

(5) $$\lim_{k \in K} x^*_{n_k}(x') = x_0^*(x') \text{ for all } x' \in X.$$

From (4) and (5) with $x' = x$, we derive that

(6) $$\lim_{k \in K}\sup\{x^*_{n_k}(y') : y' \in Y\} = x_0^*(x) - s.$$

By (6) and (5) with $x' = y \in Y$, it follows that

$$x_0^*(y) = \lim_{k\in K} x_{n_k}^*(y) \leqslant \lim_{k\in K}\sup \{x_{n_k}^*(y') : y' \in Y\} = x_0^*(x) - s.$$

Therefore, $\sup \{x_0^*(y) : y \in Y\} \leqslant x_0^*(x) - s$ or, equivalently,

$$(7) \qquad s \leqslant x_0^*(x) - \sup \{x_0^*(y) : y \in Y\}.$$

Taking into account the definition of s, relation (3) is a consequence of (7) and Theorem 2.3.

To finish the proof, we have to show that $\|x_0^*\| = 1$. Since $x \notin \bar{Y}$, it follows that $s = i > 0$ and, by (3), $x_0^* \neq 0$. Supposing that $\|x_0^*\| < 1$ (we know that $x_0^* \in B^*$, i.e., $\|x_0^*\| \leqslant 1$), we have $\lambda \cdot x_0^* \in B^*$, where $\lambda = \|x_0^*\|^{-1} > 1$, and

$$s \geqslant (\lambda \cdot x_0^*)(x) - \sup \{(\lambda \cdot x_0^*)(y) : y \in Y\} = \lambda \cdot s > s,$$

which is a contradiction. The proof of Theorem 2.4 is complete. ∎

If Y is a subset of a normed space X and $x \in X$, then a *projection* of x onto Y (or a *best approximation element* of x in Y) is an element $y \in Y$ such that $\|x - y\| \leqslant \|x - y'\|$ for all $y' \in Y$. From Theorem 2.4, one can derive a characterization of projections onto p-convex sets:

COROLLARY 2.5. *In order that y be a projection of x onto Y it is sufficient, and if Y is p-convex, also necessary to exist $x_0^* \in X^*$ with the properties: a) $\|x_0^*\| = 1$; b) $x_0^*(x - y) = \|x - y\|$; c) $x_0^*(y) = \sup \{x_0^*(y') : y' \in Y\}$.*

Proof. Suppose that $x_0^* \in X^*$ verifies conditions *a*), *b*) and *c*). Then

$$\|x - y\| = x_0^*(x - y) = x_0^*(x - y') + [x_0^*(y') - x_0^*(y)] \leqslant x_0^*(x - y') \leqslant$$

$$\leqslant \|x_0^*\| \, \|x - y'\| = \|x - y'\| \text{ for all } y' \text{ in } Y,$$

which shows that y is a projection of x onto Y.

Conversely, suppose that Y is p-convex and let y be a projection of x onto Y. The functional x_0^* in Theorem 2.4 verifies that $\|x_0^*\| = 1$ and

$$(8) \qquad \|x - y\| = x_0^*(x) - \sup \{x_0^*(y') : y' \in Y\}.$$

It remains to show that $x_0^*(y) = \sup \{x_0^*(y') : y' \in Y\}$. Otherwise, $x_0^*(y) < \sup \{x_0^*(y') : y' \in Y\}$, we have

$$\|x - y\| = \|x_0^*\| \, \|x - y\| \geqslant x_0^*(x - y) = x_0^*(x) - x_0^*(y) >$$

$$> x_0^*(x) - \sup\{x_0^*(y') : y' \in Y\},$$

which contradicts relation (8). ∎

From Corollary 2.5, one can obtain a well-known characterization of projection onto convex subsets of Hilbert spaces:

COROLLARY 2.6. *Let X be a Hilbert space, $Y \subset X$, $x \in X \setminus \bar{Y}$ and $y \in Y$. In order that y be a projection of x onto Y, it is sufficient and, if Y is p-convex, also necessary that $(x - y \,|\, y' - y) \leqslant 0$ for all y' in Y.*

Proof. Sufficiency. We have

$$\|x - y'\|^2 = (x - y' | x - y') = (x - y + y - y' | x - y + y - y') =$$

$$= \|x - y\|^2 + 2(x - y | y - y') + \|y - y'\|^2 \geqslant \|x - y\|^2$$

for all y' in Y, which shows that y is a projection of x onto Y.

Necessity. If y is a projection of x onto Y, there exists $x_0^* \in X^*$ verifying conditions *a*), *b*) and *c*) in Corollary 2.5. By Riesz's representation theorem, there is a u in X such that $\|u\| = \|x_0^*\| = 1$ and $x_0^*(z) = (z|u)$ for all $z \in X$. Then $(x - y|u) = x_0^*(x - y) = \|x - y\| = \|x - y\| \, \|u\|$, which shows that in the Schwarz inequality one has the equality sign. Consequently, there is an $\alpha \in R$ such that $u = \alpha(x - y)$. Since $\|x - y\| = (x - y|u) = (x - y|\alpha(x - y))$, it follows that $\alpha = \|x - y\|^{-1}$. Condition *c*) in Corollary 2.5 gives $(y - y'|u) \geqslant 0$ for all y' in Y, hence $(y - y' | \, \|x - y\|^{-1} \cdot (x - y)) \geqslant 0$, which implies that $(x - y|y' - y) \leqslant 0$ for all y' in Y. ∎

Directly (i.e., without appealing to Corollary 2.5), one can prove a slightly more general result:

PROPOSITION 2.7. *Let X be a pre-Hilbert space, $Y \subset X$, $x \in X \setminus \bar{Y}$, and $y \in Y$. In order that y be a projection of x onto Y it is sufficient and, if Y is p-convex, also necessary that $(x - y|y' - y) \leqslant 0$ for all y' in Y.*

Proof. The proof of the sufficiency part is the same as for Corollary 2.6. To prove the necessity, we first remark that $p^n y' + (1 - p^n) y'' \in Y$ for all $y', y'' \in Y$ and all $n \in N$. Indeed, the property is true for $n = 1$ because Y is p-convex. Assuming that $p^k y' + (1 - p^k) y'' \in Y$ for a k in N, we derive that $p^{k+1} y' + (1 - p^{k+1}) y'' = p(p^k y' + (1 - p^k) y'') + (1 - p) y'' \in Y$. Therefore $p^n y' + (1 - p^n) y'' \in Y$ for all $n \in N$.

Now, if y is a projection of x onto Y and $y \in Y$, then $p^n y' + (1 - p^n) y \in Y$ implies that

$$\|x - y\|^2 \leqslant \|x - p^n y' - (1 - p^n) y\|^2 = \|x - y - p^n(y' - y)\|^2 =$$

$$= \|x - y\|^2 - 2p^n(x - y|y' - y) + p^{2n}\|y' - y\|^2.$$

Therefore, $-2p^n(x - y|y' - y) + p^{2n}\|y' - y\|^2 \geqslant 0$ or $-2(x - y|y' - y) + p^n\|y' - y\|^2 \geqslant 0$. Taking $n \to \infty$, one obtains $(x - y|y' - y) \leqslant 0$. ∎

The next example shows that the p-convexity of Y is essential for the validity of the necessity part of Corollary 2.5.

EXAMPLE 2.8. Let $X = R$, $Y = \{-1, 1\}$, $x = 0$, $y = 1$. Then $\inf\{|x - y'| : y' \in Y\} = |x - y| = 1$. Suppose there exists $x_0^* \in R^* = R$ verifying conditions *a*), *b*) and *c*) in Corollary 2.5. Then $|x_0^*| = 1$ and $1 = |x - y| = x_0^*(-1)$, giving the contradiction

$$-1 \cdot 1 = x_0^*(1) = \sup\{(-1)y' : y' \in Y\} = 1.$$

REMARK 2.9. In the case of the convex set Y, Theorems 2.3 and 2.4 were proved by N. P. Korneĭchuk [12], pp. 28–33, and Corollary 2.5 by G. Sh. Rubinshteĭn [18] (see also A. L. Garkavi [7]).

3. Duality relations for best approximation with conical restrictions. Let X be a vector space. A *cone* in X is a nonvoid subset K of X such that $\lambda \cdot K \subset K$ for all $\lambda \geq 0$. If Y is a subset of a normed space X and K is a cone in X, denote by $d_K(.\,, Y) : X \to [0, \infty]$ the *distance function* defined by

$$(9)\qquad d_K(x, Y) = \inf\{\|x - y\| : y \in Y \text{ and } y - x \in K\} = \\ = \inf\{\|-k\| : k \in K \text{ and } x + k \in Y\},\ x \in X;$$

$d_K(x, Y)$ is called the *best approximation of x with conical restriction K by elements in Y* (we adopt the convention $\inf \emptyset = \infty$).

The problem of the existence of an y_0 in Y with $y_0 - x \in K$ such that $d_K(x, Y) = \|x - y_0\|$ contains as particular cases many approximation problems with restrictions such as the one-side approximation, i.e., the approximation of a function x by functions u satisfying $u(t) \geq x(t)$ (or $u(t) \leq x(t)$) for all t in a given interval. A comprehensive study of these problems is done in [13], Chap. II, where the duality relations are systematically applied to obtain exact solutions for various approximation problems with restrictions (especially with respect to an integral metric) for some concrete classes of functions; the considered approximating functions run the subspace of algebraic or trigonometric polynomials, the space of spline functions, and the set of functions having a degree of smoothness higher than the approximated function.

In the duality theorem proven below, we suppose that Y is a p-convex set and K is a convex cone. The following proposition shows that we gain nothing in generality supposing the cone K only p-convex.

PROPOSITION 3.1. *Every p-convex cone is convex.*

Proof. Let K be a cone in a vector space X. Then K is convex if and only if $K + K \subset K$. Hence, supposing that K is p-convex, we have to prove $K + K \subset K$. If x and y are in K, then $p^{-1}x$ and $(1 - p)^{-1}y$ are in K too (recall that $0 < p < 1$) and, therefore, $x + y = pp^{-1}x + (1 - p)(1 - p)^{-1}y \in K$. ∎

Concerning the distance function $d_K(.\,, Y)$, we prove

PROPOSITION 3.2. *Let X be a normed space. If K is a convex cone in X and Y is a nonvoid p-convex subset of X, then the distance function $d_K(.\,, Y)$ is p-convex, i.e.,*

$$(10)\qquad d_K(px + (1 - p)\,x', Y) \leq p d_K(x, Y) + (1 - p)\, d_K(x', Y)$$

for all x and x' in X.

Proof. It is sufficient to prove (10) when $d_K(x, Y)$ and $d_K(x', Y)$ are finite numbers. Given $\varepsilon > 0$, there exist y and y' in Y such that $y - x$, $y' - x' \in K$, $\|x - y\| < d_K(x, Y) + \varepsilon$ and $\|x' - y'\| < d_K(x, Y) + \varepsilon$. The p-convexity of Y and the convexity of K imply $py + (1 - p)y' \in Y$ and $py + (1 - p)y' - (px + (1 - p)\,x') = p(y - x) + (1 - p)(y' - x') \in K$, so that $d_K(px + (1 - p)\,x', Y) \leq \|px + (1 - p)\,(x' - (py + (1 - p)\,y')\| \leq p\|x - y\| + (1 - p)\|x' - y'\| < pd_K(x, Y) + (1 - p)d_K(x', Y) + \varepsilon$. As $\varepsilon > 0$ is arbitrary, inequality (10) holds. ∎

Let X be a normed space and let K be a cone in X. Given a subspace Z of X, denote by $Z^{\cdot}$ the algebraic dual of Z. For $z^{\cdot}$ in $Z^{\cdot}$ we put

$$(11)\qquad |||z^{\cdot}||| = \sup\{z^{\cdot}(z) : z \in Z \cap (-K) \text{ and } \|z\| \leq 1\}$$

(the case $|||z^{\cdot}||| = \infty$ is not excluded). It is easily seen that if $|||z^{\cdot}||| < \infty$, then

$$(12)\qquad z^{\cdot}(z) \leq |||z^{\cdot}|||\ \|z\| \text{ for all } z \text{ in } Z \cap (-K).$$

Also, if $z^* \in Z^*$, i.e., z^* is a continuous linear functional on Z, then

$$(13)\qquad |||z^*||| \leq \|z^*\|, \text{ where } \|z^*\| = \sup\{|z^*(z)| : z \in Z,\ \|z\| \leq 1\}.$$

Denote also

$$(14)\qquad B_Z^{\cdot} = \{z^{\cdot} \in Z^{\cdot} : |||z^{\cdot}||| \leq 1\}.$$

Now, we are in position to state the main result of this section:

THEOREM 3.3. *Let X be a normed space, let K be a convex cone in X, let Z be a subspace of X and let Y be a p-convex subset of Z such that $(x + K) \cap Y \neq \emptyset$ for all $x \in Z$. If the distance function $d_K(.\,, Y)$ is continuous at a least one point in Z relatively to Z, then the duality relation*

$$(15)\qquad d_K(x, Y) = \sup\{z^{\cdot}(x) - \sup\{z^{\cdot}(y) : y \in Y\} : z^{\cdot} \in B_Z^{\cdot}\}$$

holds for all x in Z. If, moreover, $x \in Z \setminus Y$, then there exists $z_0^* \in Z^*$ with $|||z_0^*||| = 1$ such that the first supremum in the right side of (15) is achieved at z_0^*, i.e.,

$$d_K(x, Y) = z_0^*(x) - \sup\{z_0^*(y) : y \in Y\}.$$

Proof. For $x \in Z$, put $E(x) = d_K(x, Y)$ and

$$(16)\qquad S(x) = \sup\{z^{\cdot}(x) - \sup\{z^{\cdot}(y) : y \in Y\} : z^{\cdot} \in B_Z^{\cdot}\} = \\ = \sup\{\inf\{z^{\cdot}(x - y) : y \in Y\} : z^{\cdot} \in B_Z^{\cdot}\}.$$

First, we shall show that

$$(17)\qquad S(x) \leq E(x).$$

By (9), $E(x) = \inf\{\|-k\| : k \in K,\ x + k \in Y\}$, and $k = x + k - x \in Z$ for all $k \in K$ such that $x + k \in Y \subset Z$. Therefore, $-k \in Z$ and taking into account (12), one obtains

$$\inf\{z^{\cdot}(x - y) : y \in Y\} \leq z^{\cdot}(x - x - k) \leq |||z^{\cdot}|||\ \|-k\| \leq \|-k\|$$

for all $z^{\cdot} \in B_Z^{\cdot}$. Taking the infimum with respect to all k in K such that $x + k \in Y$, it follows that $\inf\{z^{\cdot}(x - y) : y \in Y\} \leq E(x)$, so that $S(x) = \sup\{\inf\{z^{\cdot}(x - y) : y \in Y\} : z^{\cdot} \in B_Z^{\cdot}\} \leq E(x)$.

Denote by epi E the epigraph of the function E, i.e.,

$$\text{epi } E = \{(z, \alpha) \in Z \times R : E(z) \leq \alpha\}.$$

Since E is p-convex (Proposition 3.2), its epigraph is a p-convex subset of $Z \times R$. By the hypotheses of the theorem, there is a point z_0 in Z at which E is continuous. We shall show that $(z_0, E(z_0) + 1)$ is an interior point of epi E. To this end, by the continuity condition, there exists a $\delta > 0$ such that $|E(z) - E(z_0)| < 1/2$ for all z in Z with $\|z - z_0\| < \delta$. Remark that the neighbourhood $\{z \in Z : \|z - z_0\| < \delta\} \times]E(z_0) + \frac{1}{2}, \infty[$ of the point $(z_0, E(z_0) + 1)$ is included in epi E, so that $(z_0, E(z_0) + 1)$ will be in the interior of epi E. Indeed, if $(z, \alpha) \in Z \times R$ is such that $\|z - z_0\| < \delta$ and $\alpha > E(z_0) + \frac{1}{2}$, then $E(z) - E(z_0) < \frac{1}{2}$ implies that

$$\alpha > E(z_0) + \frac{1}{2} > E(z_0) + E(z) - E(z_0) = E(z),$$

hence $(z, \alpha) \in$ epi E.

The point $(x, E(x))$ is a boundary point of epi E because $(x, E(x)) \in$ epi E, and if

$$V = \{z \in Z : \|x - z\| < r\} \times]E(x) - \varepsilon, E(x) + \varepsilon[, \ r > 0, \ \varepsilon > 0,$$

is a neighbourhood of $(x, E(x))$ in $Z \times R$, then $\left(x, E(x) - \frac{\varepsilon}{2}\right) \in V \setminus$ epi E. Applying Theorem 1.3, there exists a closed hyperplane in $Z \times R$ supporting epi E at the point $(x, E(x))$. This means that there is $(z^*, \lambda) \in Z^* \times R = (Z \times R)^*$, $(z^*, \lambda) \neq (0, 0)$ such that

$$z^*(x) + \lambda \cdot E(x) \geqslant z^*(z) + \lambda \cdot \alpha \tag{18}$$

for all $z \in Z$ and all $\alpha \in R$ with $\alpha \geqslant E(z)$.

If $\lambda = 0$, then $z^*(x) \geqslant z^*(z)$ for all $z \in Z$, implying that $z^* = 0$, which contradicts the hypothesis that $(z^*, \lambda) \neq (0, 0)$. Therefore, $\lambda \neq 0$ and taking $z = x$ in (18), one obtains

$$\lambda \cdot E(x) \geqslant \lambda \cdot \alpha \Leftrightarrow \lambda \cdot [E(x) - \alpha] \geqslant 0 \quad \text{for all} \quad \alpha \geqslant E(x),$$

which implies $\lambda \leqslant 0$. Dividing inequality (18) by $-\lambda > 0$ and denoting $z_0^* = -\lambda^{-1} \cdot z^*$, one obtains

$$z_0^*(x) - E(x) \geqslant z_0^*(z) - \alpha \tag{19}$$

for all $z \in Z$ and all $\alpha \in R$ with $\alpha \geqslant E(z)$. When $\alpha = E(z)$, inequality (19) becomes

$$z_0^*(x) - E(x) \geqslant z_0^*(z) - E(z) \quad \text{for all} \ z \in Z. \tag{20}$$

To conclude the proof of Theorem 3.3, we need the following lemma which appears in [13], p. 38, but our proof differs from the one given therein.

LEMMA 3.4. *If $z^\cdot \in Z^\cdot$ satisfies $|||z^\cdot||| \leqslant 1$, then*

$$\sup\{z^\cdot(z) - E(z) : z \in Z\} = \sup\{z^\cdot(y) : y \in Y\}. \tag{21}$$

If $|||z^\cdot||| > 1$ (including the case $|||z^\cdot||| = \infty$), then

$$\sup\{z^\cdot(z) - E(z) : z \in Z\} = \infty. \tag{22}$$

Proof of Lemma 3.4. Let $z^\cdot \in Z^\cdot$ with $|||z^\cdot||| \leqslant 1$, and let $y \in Y$. From $0 \in K$ and $y + 0 = y \in Y$, it follows that

$$0 \leqslant E(y) = \inf\{\|y - (y + k)\| : \ k \in K, \ y + k \in Y\} \leqslant \|y - y\| = 0$$

which yields $E(y) = 0$ for all $y \in Y$. This equality and the inclusion $Y \subset Z$ produce

$$\sup\{z^\cdot(z) - E(z) : z \in Z\} \geqslant \sup\{z^\cdot(y) - E(y) : y \in Y\} = \sup\{z^\cdot(y) : y \in Y\}.$$

Now, taking into account definition (16) of S and inequality (17), one obtains

$$z^\cdot(z) - \sup\{z^\cdot(y) : y \in Y\} \leqslant S(z) \leqslant E(z) \ \text{for all} \ z \in Z$$

giving the opposite inequality

$$\sup\{z^\cdot(z) - E(z) : z \in Z\} \leqslant \sup\{z^\cdot(y) : y \in Y\},$$

needed for the proof of equality (21).

If $|||z^\cdot||| > 1$, then there exists a $k \in Z \cap (-K)$ with $\|k\| = 1$ such that $z^\cdot(k) = 1 + \alpha$, where $\alpha > 0$. Since $k \in Z$, it follows (by the hypotheses of Theorem 3.3) that $Y \cap (k + K) \neq \emptyset$, so that there are $k' \in K$ and $y_0 \in Y$ such that $y_0 = k + k'$ or $y_0 - k = k' \in K$. For any $\lambda \geqslant 1$, the relation $(\lambda - 1) \cdot (-k) \in K$ implies that $y_0 + \lambda(-k) = y_0 - k + (\lambda - 1)(-k) \in K + K \subset K$ (see the proof of Proposition 3.1). Therefore,

$$E(\lambda k) = \inf\{\|\lambda k - y\| : y \in Y, \ y - \lambda k \in K\} \leqslant \|\lambda k - y_0\| \leqslant \lambda\|k\| + \|y_0\| = \lambda + \|y_0\|$$

for all $\lambda \geqslant 1$. Consequently,

$$z^\cdot(+\lambda k) - E(\lambda k) = \lambda(1 + \alpha) - E(\lambda k) \geqslant \lambda(1 + \alpha) - \|y_0\| - \lambda = \lambda\alpha - \|y_0\|.$$

Since $\alpha > 0$, $\lambda k \in Z$ and $-\lambda k \in Z$ for all $\lambda \geqslant 1$, it follows that

$$\sup\{z^\cdot(z) - E(z) : z \in Z\} \geqslant \sup\{z^\cdot(+\lambda k) - E(\lambda k) : \lambda \geqslant 1\} = \infty.$$

Lemma 3.4 is proved. ∎

Now, let us continue the proof of Theorem 3.3. We intend to show that the functional z_0^* constructed above is in $B_{Z^\cdot}$, i. e., $|||z_0^*||| \leqslant 1$.

Supposing the contrary, $|||z_0^*||| > 1$, and using Lemma 3.4 and inequality (20), we obtain

$$z_0^*(x) - E(x) \geqslant \sup\{z_0^*(z) - E(z) : z \in Z\} = \infty.$$

On the other hand, by hypotheses of Theorem 3.3, $(x + K) \cap Y \neq \emptyset$, so that $E(x)$ is a finite number. The obtained contradiction shows that we must have $|||z_0^*||| \leqslant 1$, therefore relation (21) of the same Lemma and inequality (20) yield:

$$z_0^*(x) - E(x) \geqslant \sup\{z_0^*(z) - E(z) : z \in Z\} = \sup\{z_0^*(y) : y \in Y\}.$$

From this inequality and inequality (17) one derives that

$$S(x) \geqslant z_0^*(x) - \sup\{z_0^*(y) : y \in Y\} \geqslant E(x) \geqslant S(x), \tag{23}$$

hence,

$$E(x) = S(x) = z_0^*(x) - \sup\{z_0^*(y) : y \in Y\}. \tag{24}$$

To conclude the proof, we have to show that if $x \in Z \setminus \bar{Y}$, then $|||z_0^*||| = 1$. By (24), $z_0^* \neq 0$ because $x \notin \bar{Y}$ implies that $E(x) > 0$. We know that $|||z_0^*||| \leqslant 1$. If $|||z_0^*||| < 1$, then $|||\lambda z_0||| = 1$, where $\lambda = |||z_0|||^{-1} > 1$, and reasoning like in the final part of the proof of Theorem 2.4 we get a contradiction.

The proof of Theorem 3.3. is complete. ▮

REMARK 3.5. When $Z = K = X$, the distance function $d_K(\cdot, Y)$ agrees with the usual distance function $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$, $x \in X$, and, as it is well known, this function is continuous (in fact it is even Lipschitz, i.e., $|d(x, Y) - d(x', Y)| \leqslant \|x - x'\|$ for any x, x' in X, see [20], p. 391). Therefore, Theorem 3.3 extends Theorems 2.3 and 2.4. The functional $d_K(\cdot, Y)$ is not always continuous as is shown by an example in [13], p. 10.

The following example shows that there exist p-convex functions defined on p-convex sets which are not continuous on the whole domain of definition.

EXAMPLE 3.6. Let $X = R^2$ equipped with the Euclidean norm and let

$$Y = \{(x, y) \in R^2 : |x| + |y| < 1\} \cup \{(x, y) \in Q^2 : |x| + |y| = 1\},$$

where Q denotes the set of rational numbers. The function $f : Y \to R$, defined by $f(x, y) = |x| + |y|$ for $|x| + |y| < 1$, and $f(x, y) = 2$ for $(x, y) \in Q^2$ with $|x| + |y| = 1$, is $\frac{1}{2}$-convex but it is continuous only on $\operatorname{int} Y = \{(x, y) \in R^2 : |x| + |y| < 1\}$.

Like in the case of best approximation by elements of a p-convex set (Corollary 2.5), from Theorem 3.3 one can derive a characterization of elements of best approximation with conical restrictions.

COROLLARY 3.7. *Let X be a normed space, let K be a convex cone in X, let Z be a subspace of X, $x \in Z \setminus \bar{Y}$ and $y \in Y \cap (x + K)$, where Y is a subset of Z such that $Y \cap (z + K) \neq \emptyset$ for all $z \in Z$. In order that y be a projection of x onto $Y \cap (x + K)$, it is sufficient and, if Y is p-convex, also necessary to exist $z_0^{\cdot} \in Z^{\cdot}$ with the properties : a) $|||z_0^{\cdot}||| = 1$; b) $z_0^{\cdot}(x - y) = \|x - y\|$; and c) $z_0^{\cdot}(y) = \sup\{z_0^{\cdot}(y') : y' \in Y\}$. If Y is p-convex, then the functional $z_0^{\cdot}$ can be chosen to be continuous on Z.*

Proof. Let $z_0^{\cdot}$ be a functional in $Z^{\cdot}$ satisfying a), b) and c). For every $y' \in Y$ with $y' \in x + K$ the inequality (12) implies

$$\|x - y\| = z_0^{\cdot}(x - y) = z_0^{\cdot}(x - y') + z_0^{\cdot}(y') - z_0^{\cdot}(y) \leqslant$$
$$\leqslant z_0^{\cdot}(x - y') \leqslant |||z_0^{\cdot}||| \; \|x - y'\| = \|x - y'\|,$$

which shows that y is a projection of x onto $Y \cap (x + K)$.

Conversely, suppose that Y is p-convex and let y be a projection of x onto $Y \cap (x + K)$. By Theorem 3.3, there exists $z_0^* \in Z^*$, with $|||z_0^*||| = 1$, such that

$$\|x - y\| = z_0^*(x) - \sup\{z_0^*(y') : y' \in Y\}. \tag{25}$$

The proof will be complete if we show that $\sup\{z_0^*(y') : y' \in Y\} = z_0^*(y)$. Otherwise, $\sup\{z_0^*(y') : y' \in Y\} > z_0^*(y)$ and, since $x - y \in (-K) \cap Z$, inequality (12) yields

$$\|x - y\| = |||z_0^*||| \; \|x - y\| \geqslant z_0^*(x) - z_0^*(y) > z_0^*(x) -$$
$$- \sup\{z_0^*(y') : y' \in Y\},$$

contradicting equality (25). ▮

4. Best approximation by elements of caverns. A subset Y of a normed space X is called *p-cavern* if its complement $X \setminus Y$ is a bounded p-convex set with nonvoid interior. The study of best approximation by elements of caverns (subsets of a normed space with nonvoid bounded convex and open complement) was done by C. Franchetti and I. Singer [6]. The problem of best approximation by elements of caverns was posed by V. Klee [10] (see also [11]) in connection with the still unsolved problem of convexity of Chebyshev sets in Hilbert spaces. The term "Klee cavern" was proposed by E. Asplund [2].

The following theorem extends to p-caverns the main duality result in [6], Theorem 2.1.

THEOREM 4.1. *Let X be a normed space, let Y be a p-cavern in X and $x \in X \setminus Y$. Then*

$$\inf\{\|x - y\| : y \in Y\} = \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - \tag{26}$$
$$- x^*(x) : x^* \in S^*\},$$

where $S^* = \{x^* \in X^* : \|x^*\| = 1\}$ is the unit sphere in the dual space X^* of X.

Proof. Put $d = \inf\{\|x - y\| : y \in Y\}$ and $I = \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) : x^* \in S^*\}$. Let $x^* \in S^*$, and denote $c = \sup\{x^*(x') : x' \in X \setminus Y\}$ (c is finite because $X \setminus Y$ is nonvoid and bounded).

The hyperplane $H = \{x' \in X : x^*(x') = c\}$ is included in $\bar{Y}$. Indeed, if $x' \in X \setminus \bar{Y}$, then $x' \in \operatorname{int}(X \setminus Y)$. Since $X \setminus Y \subset \{x'' \in X : x^*(x'') \leqslant c\}$, it follows that $\operatorname{int}(X \setminus Y) \subset \operatorname{int}\{x'' \in X : x^*(x'') \leqslant c\} = \{x'' \in X : x^*(x'') < c\}$, so that $x^*(x') < c$. Therefore, $x' \notin H$ showing that $H \subset \bar{Y}$.

By Ascoli's formula for the distance from a point to a hyperplane in a normed space (see [20], p. 24) we have

$$d = d(x, Y) = d(x, \bar{Y}) = \inf\{\|x - y\| : y \in \bar{Y}\} \leqslant \inf\{\|x - y\| : y \in H\} =$$
$$= |x^*(x) - c| / \|x^*\| = \sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x).$$

Therefore,

$$d \leqslant \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) : x^* \in S^*\} = I. \tag{27}$$

Denoting by fr Y the boundary of Y, we shall show that

(28) $$d = \inf\{\|x - y\| : y \in Y\} = \inf\{\|x - x'\| : x' \in \mathrm{fr}\, Y\}.$$

Put $d' = \inf\{\|x - x'\| : x' \in \mathrm{fr}\, Y\}$. The inclusion $\mathrm{fr}\, Y \subset \bar{Y}$ implies that $d' \geqslant d(x, \bar{Y}) = d(x, Y) = d$. Supposing that $d < d'$, there is an y in Y such that $\|x - y\| < d'$. Let $t_0 = \inf\{t \in [0, 1] : x + t(y - x) \in Y\}$. Then $x_0 = x + t_0(y - x)$ is a boundary point of Y and $\|x - x_0\| = t_0\|y - x\| \leqslant \leqslant \|y - x\| < d'$, which contradicts the definition of d'. Consequently, $d = d'$ and (28) is proven.

To prove the opposite inequality of (27), let y' be an element of $\mathrm{fr}\, Y = \mathrm{fr}(X \setminus Y)$. By Theorem 1.3, there exists $x^* \in S^*$ such that $x^*(x') \leqslant \leqslant x^*(y')$ for all $x' \in X \setminus Y$. It follows that

$$I = \inf\{\sup\{y^*(x') : x' \in X \setminus Y\} - y^*(x) : y^* \in S^*\} \leqslant$$
$$\leqslant \sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) \leqslant x^*(y') - x^*(x) \leqslant$$
$$\leqslant \|x^*\|\, \|y' - x\| = \|y' - x\|.$$

From this and (28), one obtains

(29) $$I \leqslant \inf\{\|x - y'\| : y' \in \mathrm{fr}\, Y\} = d.$$

Inequalities (27) and (29) imply that $I = d$, and Theorem 4.1 is proved. ∎

Like in the preceding sections, we derive from Theorem 4.1 a characterization of projections onto p-caverns. The next corollary is analogous to Theorem 3.1 in [6].

COROLLARY 4.2. *Let X be a normed space, let Y be a p-cavern in X, $x \in X \setminus Y$ and $y \in Y$. In order that y be a projection of x onto Y it is necessary and sufficient that $y \in \mathrm{fr}\, Y$ and there exists a functional $x_0^* \in S^*$ verifying the conditions :*

a) $$\sup\{x_0^*(x') : x' \in X \setminus Y\} - x_0^*(x) = \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) : x^* \in S^*\};$$

b) $$x_0^*(y) = \sup\{x_0^*(x') : x' \in X \setminus Y\};$$

c) $$x_0^*(y - x) = \|y - x\|.$$

Proof. Admit that y is a projection of x onto Y. Then $y \in \mathrm{fr}\, Y = \mathrm{fr}\,(X \setminus Y)$ and, by Theorem 1.3, there exists $x_0^* \in S^*$ such that $x_0^*(y) = \sup\{x_0^*(x') : x' \in X \setminus Y\}$ which shows that equality *b)* is true. Taking into account the duality relation (26), one obtains

$$d = \|y - x\| \geqslant x_0^*(y) - x_0^*(x) = \sup\{x_0^*(x') : x' \in X \setminus Y\} - x_0^*(x) \geqslant$$
$$\geqslant \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) : x^* \in S^*\} = d,$$

which shows that relations *a)* and *c)* are also true.

Conversely, suppose $y \in \mathrm{fr}\, Y$ and $x_0^* \in S^*$ verifies conditions *a)*, *b)* and *c)* from the Corollary 4.2. Appealing again to the duality relation (26), it follows that

$$\|y - x\| = x_0^*(y) - x_0^*(x) = \sup\{x_0^*(x') : x' \in X \setminus Y\} - x_0^*(x) =$$
$$= \inf\{\sup\{x^*(x') : x' \in X \setminus Y\} - x^*(x) : x^* \in S^*\} = d,$$

showing that y is a projection of x onto Y. ∎

REMARK 4.3. By Corollary 4.2, it results that if y is a projection of x onto Y, then there exists a functional $x_0^* \in S^*$ such that the infimum I in the duality relation (26) is attained at x_0^*. The converse of this assertion is not true as was shown in [6], i. e., the existence of a functional x_0^* at which the infimum I in the right side of (26) is attained does not imply the existence of a best approximation element of x in Y. In the same paper an example was given of a cavern Y in l^2 and an element x of l^2 having no best approximation element in Y, and such that the infimum in the right side of (26) is not attained.

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