MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 16, N° 2, 1987, pp. 117—126

THEOREM OF MOTZKIN'S ALTERNATIVE FOR NONHOMOGENEOUS COMPLEX LINEAR EQUATIONS AND INEQUALITIES

DOREL I. DUCA
(Cluj-Napoca)

Abstract. The classical theorems of the alternative of Motzkin, Gordan and others are extended to nonhomogeneous complex linear equations and inequalities.

0. Introduction. The theorems of the alternative play an important role in establishing of necessary conditions for optimal solutions of a mathematical programming problem and necessary conditions for efficient solutions of a vectorial programming problem.

The extension of mathematical programming theory to complex space necessarily request the extension of theorems of the alternative to complex space. To show that the duality theory of linear programming in real space also holds in complex space, Ben-Israel [2] has proved the following extension of Farkas theorem to complex space:

THEOREM 0. *Let* $A \in C^{m\times n}$, $a \in C^m$ *and let* S *be a polyhedral cone in* C^n. *Then the system*

$$\begin{cases} Az = a \\ z \in S, \end{cases}$$

is consistent, if and only if

$$A^H v \in S^* \text{ implies } \operatorname{Re}\langle a, v\rangle \geqq 0.$$

Equivalent formulations of theorem 0 have been given by several authors, among which we should mention: A. Ben-Israel [3], B. Mond and M. A. Hanson [22], [23], B. Mond [21], R. N. Kaul [18], D. I. Duca [11], [13], [14], I. M. Stancu-Minasian and D. I. Duca [25].

In 1969, A. Ben-Israel [3] extended the theorem of the alternative of Motzkin [24] for homogeneous linear equations and inequalities to complex space.

In this paper an extension of the Motzkin theorem of the alternative for nonhomogeneous linear equations and inequalities to complex space is given.

1. Notations and preliminaries. Let $C^n(R^n)$ denote the n-dimensional complex (real) vector space, $R^n_+ = \{x \in R^n : x = (x_j)$ with $x_j \geqq 0$ for all $j \in \{1, \ldots, n\}\}$ the non-negative orthant of R^n, and $C^{m\times n}(R^{m\times n})$ the set of $m \times n$ complex (real) matrices. If A is a matrix or a vector, then A^T, $\bar{A}$, A^H denotes its transpose, complex conjugate and conjugate transpose respectively.

For $z = (z_j) \in C^n$:

$\mathrm{Re}\, z = (\mathrm{Re}\, z_j) \in R^n$ denotes the *real part* of z,
$\mathrm{Im}\, z = (\mathrm{Im}\, z_j) \in R^n$ denotes the *imaginary part* of z,
$\arg z = (\arg z_j)$ denotes the *argument* of z,
$|z| = (|z_j|) \in R^n_+$ denotes the *module* of z.

For any $x = (x_j)$, $y = (y_j) \in R^n$, we consider:

$$x \leqq y \ (x < y) \text{ iff } x_j \leqq y_j \ (x_j < y_j) \text{ for all } j \in \{1, \ldots, n\},$$
$$x \leq y \text{ iff } x \leqq y \text{ and } x \neq y.$$

For $z, w \in C^n$: $\langle z, w\rangle = w^H z$ denotes the *inner product of* z and w.

A nonempty set S in C^n is a:
(*i*) ***convex cone*** if $S + S \subseteq S$ and if $r \in R_+$ implies that $rS \subseteq S$;
(*ii*) ***pointed convex*** *cone* if (*i*) and $S \cap (-S) = \{0\}$;
(*iii*) ***polyhedral cone*** if it is a finite intersection of closed half-space in C^n, each containing 0 in its boundary.

For any nonempty set S in C^n, let:
$S^* = \{v \in C^n : z \in S$ implies $\mathrm{Re}\langle z, w\rangle \geqq 0\}$ the *polar* of S, e.g. [2] and $\mathrm{int}\, S^* = \{v \in C^n : z \in S \setminus \{0\}$ implies $\mathrm{Re}\langle z, v\rangle > 0\}$ the *interior* of S^*.

If S is a nonempty set in C^n, then S^* is a closed convex cone. Since S^* coincides with the polar of the smallest closed convex cone containing S, e.g. [2], it suffices to study polars of closed convex cones.

If S is a closed convex cone in C^n, then:
(*i*) $\mathrm{int}\, S^*$ is nonempty if and only if S is pointed;
(*ii*) $\mathrm{int}\, S$ is nonempty if and only if S^* is pointed;
(*iii*) $\mathrm{int}\, S = \{v \in S : z \in S^* \setminus \{0\}$ implies $\mathrm{Re}\langle v, z\rangle > 0\}$.

If S and T are polyhedral cones in C^n and C^m respectively, then:
(*i*) $S \times T$ is a polyhedral cone in C^{n+m};
(*ii*) $(S \times T)^* = S^* \times T^*$.

2. Results. The main result is:

THEOREM 1. *Let* $A_1 \in C^{m\times n}$, $A_2 \in C^{m\times k}$, $B_1 \in C^{p\times n}$, $B_2 \in C^{p\times k}$, $D_1 \in C^{q\times n}$, $D_2 \in C^{q\times k}$, $a \in C^m$, $b \in C^p$, $d \in C^q$. *Let* T *be a polyhedral cone in* C^m *with nonempty interior, let* M *be a polyhedral cone in* C^p *and let* S *be a polyhedral cone in* C^n.

Then, the system

$$(1)\qquad \begin{cases} A_1 z + A_2 w - a \in \mathrm{int}\, T \\ B_1 z + B_2 w - b \in M \\ D_1 z + D_2 w = d \\ z \in S, \end{cases}$$

is consistent if and only if the systems:

$$(2)\qquad \begin{cases} A_1^H t + B_1^H u + D_1^H v \in S^* \\ A_2^H t + B_2^H u + D_2^H v = 0 \\ \mathrm{Re}\,[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] \leqq 0 \\ -t \in T^* \setminus \{0\} \\ -u \in M^*, \end{cases}$$

and

$$(3)\qquad \begin{cases} A_1^H t + B_1^H u + D_1^H v \in S^* \\ A_2^H t + B_2^H u + D_2^H v = 0 \\ \mathrm{Re}\,[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] < 0 \\ -t \in T^* \\ -u \in M^*, \end{cases}$$

are both inconsistent.

Proof. a) Suppose that system (1) is consistent. Then system (2) cannot have solutions, for then $0 < \mathrm{Re}\,[\langle A_1 z + A_2 w - a, -t\rangle + \langle B_1 z + B_2 w - b, -u\rangle + \langle D_1 z + D_2 w - d, -v\rangle + \langle A_1^H t + B_1^H u + D_1^H v, z\rangle + \langle A_2^H t + B_2^H u + D_2^H v, w\rangle] = \mathrm{Re}\,[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] \leqq 0$, by (1), (2) and the definitions of T^*, M^*, S^* and $\mathrm{int}\, T$. Neither can system (3) have solutions, for then $0 \leqq \mathrm{Re}\,[\langle A_1 z + A_2 w - a, -t\rangle + \langle B_1 z + B_2 w - b, -u\rangle + \langle D_1 z + D_2 w - d, -v\rangle + \langle A_1^H t + B_1^H u + D_1^H v, z\rangle + \langle A_2^H t + B_2^H u + D_2^H v, w\rangle] = \mathrm{Re}\,[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] < 0$, by (1), (3) and the definitions of T^*, M^* and S^*.

b) Suppose now that systems (2) and (3) are inconsistent. Since $\mathrm{int}\, T \neq \emptyset$, it follows that there exists an $h \in \mathrm{int}\, T$. Now, by the inconsistency of (2), we deduce

$$\begin{bmatrix} A_1^H & B_1^H & D_1^H \\ A_2^H & B_2^H & D_2^H \\ -a^H & -b^H & -d^H \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} \in S^* \times \{0\} \times (R_+ + iR) \times T^* \times M^* \text{ implies } t = 0.$$

Since $t = 0$ implies $\mathrm{Re}\left\langle \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ u \\ v \end{bmatrix} \right\rangle \geqq 0$, it follows that

$$\begin{bmatrix} A_1^H & B_1^H & D_1^H \\ A_2^H & B_2^H & D_2^H \\ -a^H & -b^H & -d^H \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} \in S^* \times \{0\} \times (R_+ + iR) \times T^* \times M^* \text{ implies } \mathrm{Re}\left\langle \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ u \\ v \end{bmatrix} \right\rangle \geqq 0.$$

By theorem 0, this is equivalent to : the system

$$(4)\quad \begin{cases} \begin{bmatrix} A_1 & A_2 & -a & -I & 0 \\ B_1 & B_2 & -b & 0 & -I \\ D_1 & D_2 & -d & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \\ r \\ x \\ y \end{bmatrix} = \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} z \\ w \\ r \\ x \\ y \end{bmatrix} \in (S^* \times \{0\} \times (R_+ + iR) \times T^* \times M^*)^*, \end{cases}$$

is consistent. Since $(S^* \times \{0\} \times (R_+ + iR) \times T^* \times M^*)^* = S \times C^k \times R_+ \times T \times M$, from (4), we deduce that the system

$$\begin{cases} A_1 z + A_2 w - ar - x = h \\ B_1 z + B_2 w - br - y = 0 \\ D_1 z + D_2 w - dr = 0 \\ z \in S \\ w \in C^k \\ r \in R_+ \\ x \in T \\ y \in M, \end{cases}$$

is consistent. System (5) gives

$$\begin{cases} A_1 z + A_2 w - ar = x + h \in T + \operatorname{int} T = \operatorname{int} T \\ B_1 z + B_2 w - br = y \in M \\ D_1 z + D_2 w - dr = 0 \\ z \in S \\ r \in R_+, \end{cases}$$

hence there exists $\begin{bmatrix} z^1 \\ w^1 \\ r^1 \end{bmatrix} \in C^n \times C^k \times C$ so that

$$(6)\quad \begin{cases} A_1 z^1 + A_2 w^1 - ar^1 \in \operatorname{int} T \\ B_1 z^1 + B_2 w^1 - br^1 \in M \\ D_1 z^1 + D_2 w^1 - dr^1 = 0 \\ z^1 \in S \\ r^1 \in R_+. \end{cases}$$

b_1) If $r^1 \neq 0$, then from (6) it follows that $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} (1/r^1)z^1 \\ (1/r^1)w^1 \end{bmatrix} \in C^n \times C^k$ is a solution to system (1), hence system (1) is consistent.

b_2) If $r^1 = 0$, then from (6) we deduce that

$$(7)\quad \begin{cases} A_1 z^1 + A_2 w^1 \in \operatorname{int} T \\ B_1 z^1 + B_2 w^1 \in M \\ D_1 z^1 + D_2 w^1 = 0 \\ z^1 \in S. \end{cases}$$

On the other hand, by the inconsistency of system (3) we have

$$\begin{bmatrix} A_1^H & B_1^H & D_1^H \\ A_2^H & B_2^H & D_2^H \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} \in S^* \times \{0\} \times T^* \times M^* \text{ implies } \operatorname{Re} \left\langle \begin{bmatrix} a \\ b \\ d \end{bmatrix}, \begin{bmatrix} t \\ u \\ v \end{bmatrix} \right\rangle \geqq 0.$$

By theorem 0, this is equivalent to : the system

$$\begin{cases} \begin{bmatrix} A_1 & A_2 & -I & 0 \\ B_1 & B_2 & 0 & -I \\ D_1 & D_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \\ x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix} \\ \begin{bmatrix} z \\ w \\ x \\ y \end{bmatrix} \in (S^* \times \{0\} \times T^* \times M^*)^* \end{cases}$$

is consistent. Then there exists $\begin{bmatrix} z^2 \\ w^2 \end{bmatrix} \in C^n \times C^k$ so that

$$(8)\quad \begin{cases} A_1 z^2 + A_2 w^2 - a \in T \\ B_1 z^2 + B_2 w^2 - b \in M \\ D_1 z^2 + D_2 w^2 = d \\ z^2 \in S. \end{cases}$$

Let us note now $z = z^1 + z^2 \in C^n$ and $w = w^1 + w^2 \in C^k$. Then
$A_1 z + A_2 w - a = (A_1 z^1 + A_2 w^1) + (A_1 z^2 + A_2 w^2 - a) \in \operatorname{int} T + T = \operatorname{int} T$,
$B_1 z + B_2 w - b = (B_1 z^1 + B_2 w^1) + (B_1 z^2 + B_2 w^2 - b) \in M + M = M$,
$D_1 z + D_2 w = (D_1 z^1 + D_2 w^1) + (D_1 z^2 + D_2 w^2) = 0 + d = d$,
$z = z^1 + z^2 \in S + S = S$,
by (7) and (8). Hence, system (1) is consistent. This completes the proof. ∎

Related results are :

THEOREM 2. *Let* $A_1 \in C^{m \times n}$, $A_2 \in C^{m \times k}$, $B_1 \in C^{p \times n}$, $B_2 \in C^{p \times k}$, $D_1 \in C^{q \times n}$, $D_2 \in C^{q \times k}$, *let* T *be a polyhedral cone in* C^m *with nonempty interior, let* M *be a polyhedral cone in* C^p *and let* S *be a polyhedral cone in* C^n.

Then, the system:

$$\begin{cases} A_1z + A_2w \in \operatorname{int} T \\ B_1z + B_2w \in M \\ D_1z + D_2w = 0 \\ z \in S, \end{cases}$$

is consistent, if and only if the system

$$\begin{cases} A_1^H t + B_1^H u + D_1^H v \in S^* \\ A_2^H t + B_2^H u + D_2^H v = 0 \\ -t \in T^* \setminus \{0\} \\ -u \in M^*, \end{cases}$$

is inconsistent.

Proof. Apply theorem 1 with $A_1 := A_1$, $A_2 := A_2$, $B_1 := B_1$, $B_2 := B_2$, $D_1 := D_1$, $D_2 := D_2$, $T := T$, $M := M$, $S := S$, $a := 0 \in C^m$, $b := 0 \in C^p$, $d := 0 \in C^q$. Since, in this case, system (3) is inconsistent ($\operatorname{Re}[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] = 0$), it follows that system (1) is consistent if and only if system (2) is inconsistent, which completes the present proof. ∎

THEOREM 3. *Let* $A_1 \in R^{m\times n}$, $A_2 \in R^{m\times k}$, $B_1 \in R^{p\times n}$, $B_2 \in R^{p\times k}$, $D_1 \in R^{q\times n}$, $D_2 \in R^{q\times k}$, $a \in R^m$, $b \in R^p$, $d \in R^q$. *Then the system*

$$\begin{cases} A_1x + A_2y - a > 0 \\ B_1x + B_2y - b \geqq 0 \\ D_1x + D_2y = d \\ x \geqq 0, \end{cases}$$

is consistent, if and only if the systems

$$\begin{cases} A_1^T t + B_1^T u + D_1^T v \geqq 0 \\ A_2^T t + B_2^T u + D_2^T v = 0 \\ \langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle \leqq 0 \text{ and} \\ t \leqslant 0, \\ u \leqq 0, \end{cases} \qquad \begin{cases} A_1^T t + B_1^T u + D_1^T v \geqq 0 \\ A_2^T t + B_2^T u + D_2^T v = 0 \\ \langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle < 0 \\ t \leqq 0, \\ u \leqq 0, \end{cases}$$

are both inconsistent.

Proof. Take everything in theorem 1 to be real with $T := R^m_+$, $M := R^p_+$, and $S := R^n_+$. ∎

3. Special cases. Theorems 1 and 2 yield, as special cases, a number of known results.

COROLLARY 1 (Duca [14]). *Let* $A \in C^{m\times n}$, $B \in C^{p\times n}$, $D \in C^{q\times n}$, $a \in C^m$, $b \in C^p$, $d \in C^q$, *let T be a polyhedral cone in* C^m *with nonempty interior, let M be a polyhedral cone in* C^p *and let S be a polyhedral cone in* C^n. *Then the system*

$$\begin{cases} Az - a \in \operatorname{int} T \\ Bz - b \in M \\ Dz = d \\ z \in S, \end{cases}$$

is consistent, if and only if the systems

$$\begin{cases} A^H t + B^H u + D^H v \in S^* \\ \operatorname{Re}[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] \leqq 0 \text{ and} \\ -t \in T^* \setminus \{0\} \\ -u \in M^*, \end{cases} \qquad \begin{cases} A^H t + B^H u + D^H v \in S^* \\ \operatorname{Re}[\langle a, t\rangle + \langle b, u\rangle + \langle d, v\rangle] < 0 \\ -t \in T^* \\ -u \in M^*, \end{cases}$$

are both inconsistent.

Proof. Apply theorem 1 with $A_1 := A$, $B_1 := B$, $D_1 := D$, $A_2 := 0 \in C^{m\times k}$, $B_2 := 0 \in C^{p\times k}$, $D_2 := 0 \in C^{q\times k}$, $a := a$, $b := b$, $d := d$, $T := T$, $M := M$, $S := S$. ∎

This result is an extension of the Motzkin theorem to nonhomogeneous complex linear equations and inequalities.

COROLLARY 2 (Ben-Israel [3]). *Let* $A \in C^{m\times n}$, $B \in C^{p\times n}$, $D \in C^{q\times n}$, *let T be a polyhedral cone in* C^m *with nonempty interior, let M be a polyhedral cone in* C^p *and let S be a polyhedral cone in* C^n. *Then the system:*

$$\begin{cases} Az \in \operatorname{int} T \\ Bz \in M \\ Dz = 0 \\ z \in S, \end{cases}$$

is consistent, if and only if the system

$$\begin{cases} A^H t + B^H u + D^H v \in S^* \\ -t \in T^* \setminus \{0\} \\ -u \in M^*, \end{cases}$$

is inconsistent.

Proof. Apply theorem 2 with $A_1 := A$, $A_2 := 0 \in C^{m\times k}$, $B_1 := B$, $B_2 := 0 \in C^{p\times k}$, $D_1 := D$, $D_2 := 0 \in C^{q\times k}$, $T := T$, $M := M$, $S := S$. ∎

This result is an extension of the Motzkin theorem to complex space.

COROLLARY 3 (Mond and Hanson [22]). *Let* $A \in C^{m\times n}$, $B \in C^{p\times n}$, $D \in C^{q\times n}$ *and* $\alpha \in R^p_+$ *with* $\alpha \leqq \frac{\pi}{2} e$, *where* $e = (1, \ldots, 1)^T \in R^p$. *Then the system*

$$\begin{cases} \operatorname{Re}(Az) > 0 \\ |\arg(Bz)| \leqq \alpha \\ Dz = 0, \end{cases}$$

is consistent, if and only if the system

$$\begin{cases} A^H t + B^H u + D^H v = 0 \\ \operatorname{Im} t = 0 \\ \operatorname{Re} t \geqq 0 \\ |\arg u| \leqq \frac{\pi}{2} e - \alpha \end{cases}$$

is inconsistent.

Proof. Apply theorem 2 with $A_1 := A$, $A_2 := 0 \in C^{m \times k}$, $B_1 := B$, $B_2 := 0 \in C^{p \times k}$, $D_1 := D$, $D_2 := 0 \in C^{q \times k}$, $T := R^m_+ + iR^m$, $S := C^n$, $M := \{u \in C^p : |\arg u| \leqq \alpha\}$. ∎

COROLLARY 4 (Duca [13]). *Let* $B_1 \in C^{p \times n}$, $B_2 \in C^{p \times k}$, $D_1 \in C^{q \times n}$, $D_2 \in C^{q \times k}$, $b \in C^p$, $d \in C^q$, *let M be a polyhedral cone in C^p and let S be a polyhedral cone in C^n. Then the system*

$$\begin{cases} B_1 z + B_2 w - b \in M \\ D_1 z + D_2 w = d \\ z \in S, \end{cases}$$

is consistent, if and only if the system

$$\begin{cases} B_1^H u + D_1^H v \in S^* \\ B_2^H u + D_2^H v = 0 \\ \operatorname{Re} [\langle b, u \rangle + \langle d, v \rangle] < 0 \\ -u \in M^*, \end{cases}$$

is inconsistent.

Proof. Apply theorem 1 with $A_1 := 0 \in C^{m \times n}$, $A_2 := 0 \in C^{m \times}$, $B_1 := B_1$, $B_2 := B_2$, $D_1 := D_1$, $D_2 := D_2$, $a := 0 \in C^m$, $b := b$, $d := d$, $T := C^m$, $M := M$, $S := S$. Since $T^* = \{0\}$, it follows that system (2) is inconsistent ($T^* \setminus \{0\} = \emptyset$). Then system (1) is consistent if and only if system (3) is inconsistent. This completes the proof. ∎

Taking $B_1 := 0 \in C^{p \times n}$, $B_2 := 0 \in C^{p \times k}$, $D_2 := 0 \in C^{q \times k}$, $b := 0 \in C^p$ and $M := C^p$ in corollary 4 we get theorem 0, the extension to complex space of Farkas theorem given by Ben-Israel [2].

Taking $S := \{z \in C : |\arg z| \leqq \alpha\}$ where $\alpha \in R^n_+$, $\alpha \leqq \frac{\pi}{2} e$, $e = (1, \ldots, 1)^T \in R^n$ in theorem 0 gives the extension to complex space of Farkas theorem given by Levinson [19].

Let $A \in C^{r \times n}$, $B \in C^{s \times n}$, $a \in C^r$, $f \in R^s$ and let L and S be polyhedral cones in C^r and C^n respectively. If

$$B_1 := \begin{bmatrix} A \\ B \end{bmatrix} \in C^{(r+s) \times n}, \quad B_2 := 0 \in C^{(r+s) \times k}, \quad D_1 := 0 \in C^{q \times n}, \quad D_2 := 0 \in C^{q \times k},$$

$d := 0 \in C^q$, $M := L \times R^s$, $S := S$, corollary 4 reduces to the extension to complex space of Farkas theorem given by Stancu-Minasian and Duca [25].

If $B_2 := 0 \in C^{p \times k}$, $D_1 := 0 \in C^{q \times n}$, $D_2 := 0 \in C^{q \times k}$, $d := 0 \in C^q$, corollary 4 reduces to the extension to complex space of Farkas theorem given by Mond [21].

The other theorems of the alternative similarly follow from the above theorems.

COROLLARY 5 (Motzkin [24]). *Let* $A_1 \in R^{m \times n}$, $A_2 \in R^{m \times k}$, $B_1 \in R^{p \times n}$, $B_2 \in R^{p \times k}$, $D_1 \in R^{q \times n}$, $D_2 \in R^{q \times k}$. *Then the system*

$$\begin{cases} A_1 x + A_2 y > 0 \\ B_1 x + B_2 y \geqq 0 \\ D_1 x + D_2 y = 0 \\ x \geqq 0, \end{cases}$$

is consistent, if and only if the system

$$\begin{cases} A_1^T t + B_1^T u + D_1^T v \geqq 0 \\ A_2^T t + B_2^T u + D_2^T v = 0 \\ t \geq 0 \\ u \geqq 0, \end{cases}$$

is inconsistent.

Proof. Take everything in theorem 2 to be real with $T := R^m_+$, $M := R^p_+$, and $S := R^n_+$.

Taking $A_2 := 0 \in R^{m \times k}$, $B_1 := 0 \in R^{p \times n}$, $B_2 := 0 \in R^{p \times k}$, $D_1 := 0 \in R^{q \times n}$, $D_2 := 0 \in R^{q \times k}$ in corollary 5 gives the transposition theorem of Gordan [16].

4. Remarks

(*i*) Theorem 1 cannot be extended to general (nonpolyhedral) closed convex cones (see [3]).

(*ii*) Theorems 0, 1, 2 are equivalent. In applications sometimes one, sometimes the other is preferred.

(*iii*) For applications of the theorems of the alternative in complex space, see, for example, [1], [2], [6], [7], [8], [9], [10], [11], [12], [17], [19], [21], [22], [23], [25].

REFERENCES

[1] Abrams, R. A. and Ben-Israel, A., *Nonlinear programming in complex space: necessary conditions*, SIAM J. Control, **9** (1971), no. 4, 606–620.

[2] Ben-Israel, A., *Linear equations and inequalities on finite-dimensional, real or complex, vector spaces: a unified theory*, J. Math. Anal. Appl., **27** (1969), no. 2, 367–389.

[3] Ben-Israel, A., *Theorems of the alternative for complex linear inequalities*, Israel J. Math., **7** (1969), no. 2, 129–136.

[4] Ben-Israel, A. and Abrams, R. A., *On the key theorems of Tucker and Levinson for complex linear inequalities*, J. Math. Anal. Appl., **29** (1970), no. 3, 640–646.

[5] Craven, B. D. and Mond, B., *A Fritz John theorem in complex space*, Bull. Austral. Math. Soc., **8** (1973), no. 2, 215–220.

[6] Craven, B. D. and Mond, B., *Real and complex Fritz John theorems*, J. Math. Anal. Appl., **44** (1973), no. 3, 773—778.
[7] Craven, B. D. and Mond, B., *On duality in complex linear programming*, J. Austral. Math. Soc., **16** (1973), 172—175.
[8] Dragomirescu, M. și Malița, M., *Programare neliniară*, Ed. științifică, București, 1972.
[9] Duca, D. I., *On vectorial programming in complex space*, Studia Univ. Babeș-Bolyai, Math., **24** (1979), no. 1, 51—56.
[10] Duca, D. I., *Necessary optimality criteria in nonlinear programming in complex space with differentiability*, L'Analyse numérique et la théorie de l'approximation, **9** (1980), no. 2, 163—179.
[11] Duca, D. I., *Mathematical programming in complex space*, Doctoral thesis, University of Cluj-Napoca, Cluj-Napoca, 1981.
[12] Duca, D. I., *Efficiency criteria in vectorial programming in complex space without convexity*, Cahiers du C.E.R.O., **26** (1984), no. 3—4, 217—226.
[13] Duca, D. I., *On the Farkas type theorem for complex linear equations and inequalities*, Itinerant seminar on functional equations, approximation and convexity (Cluj-Napoca, 1987), 143—148, Preprint, 87—6, Univ. "Babeș-Bolyai", Cluj-Napoca, 1987.
[14] Duca, D. I., *On theorems of the alternative for nonhomogeneous complex linear equations and inequalities*, Seminar on optimization theory, Preprint, 87—8, Univ. "Babeș-Bolyai", Cluj-Napoca, 1987.
[15] Farkas, J., *Über die Theorie der einfachen Ungleichungen*, J. Reine Angew. Math., **124** (1902), 1—24.
[16] Gordan, P., *Über die Auflösungen linearer Gleichungen mit reelen Coefficienten*, Math. Ann., **6** (1873), 23—28.
[17] Gulati, T. R., *A Fritz John type sufficient optimality theorem in complex space*, Bull. Austral. Math. Soc., **11** (1974), no. 2, 219—224.
[18] Kaul, R. N., *On linear inequalities in complex space*, Amer. Math. Monthly, **77** (1970), no. 9, 956—960.
[19] Levinson, N., *Linear programming in complex space*, J. Math. Anal. Appl., **14** (1966), no. 1, 44—62.
[20] Mangasarian, O. L., *Nonlinear programming*, McGraw Hill, New York, 1969.
[21] Mond, B., *An extension of the transposition theorems of Farkas and Eisenberg* J. Math. Anal. Appl., **32** (1970), no. 3, 559—566.
[22] Mond, B. and Hanson, M. A., *A complex transposition theorem with applications to complex programming*, Linear Algebra and Appl., **2** (1969), 49—56.
[23] Mond, B. and Hanson, M. A., *Some generalizations and applications of a complex transposition theorem*, Linear Algebra and Appl., **2**(1969), 401—411.
[24] Motzkin, T. S., *Beiträge zur Theorie der linearen Ungleichiben*, Inaugural Dissertation, Basel 1933; Jerusalem, Azriel, 1936 (English translation: U.S. Air Force, Project Rand, Report T—22, 1952).
[25] Stancu-Minasian, I. M. and Duca, D. I., *Multiple objective linear fractional optimization in complex space* (to appear).

Received 30.IV.1987

University of Cluj-Napoca
Faculty of Mathematics
3400 Cluj-Napoca
România