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PSEUDO-GEOMETRIC INEQUALITIES

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In the present paper we introduce the so-called pseudo-geometric inequality which represents a generalization of the abstract geometric inequality introduced by Duffin, Peterson and Zener in [3]. Then we construct the pseudo-geometric inequalities (2) and (9) (Theorems 1 and 3) and we demonstrate that these inequalities are not abstract geometric inequalities (Theorems 2 and 4). From theorems 2 and 4, we see that the duality theory developed in [1] is not a particular case of the duality theory developed in [3].

DEFINITION 1. *An inequality is said to be a pseudo-geometric inequality if it satisfies the following postulates:*

(*i*) *The inequality is a scalar product inequality of the form:*

$$\sum_{i=1}^{n} x_i y_i \leqslant \lambda(y) G(x) - F(y), \tag{1}$$

which is valid for each vector $x = (x_1, \ldots, x_n)$ *in an open convex set* $C \subseteq R^n$ *and each vector* $y = (y_1, \ldots, y_n)$ *in a cone* $K \subseteq R^n$, *where* F, $\lambda : K \to R$ *and* $G : C \to R$ *are functions.*

(*ii*) *The function* λ *is nonnegative on the cone* K.

(*iii*) *The function* G *is differentiable on the open convex set* C.

In [3] Duffin, Peterson and Zener introduced the so-called abstract geometric inequality.

DEFINITION 2. *An inequality is said to be an abstract geometric inequality if it satisfies the following postulates:*

(*i*) *The inequality is a scalar product inequality of the form* (1), *which is valid for each vector* $x = (x_1, \ldots, x_n)$ *in an open convex set* $C \subseteq R^n$ *and each vector* $y = (y_1, \ldots, y_n)$ *in a cone* $K \subseteq R^n$, *where* F, $\lambda : K \to R$ *and* $G : C \to R$ *are functions.*

(*ii*) *For any vector* x *in* C *there is a nonzero vector* z *in* K *such that inequality* (1) *becomes an equality for each vector* y *on the ray emanating from the origin through the point* z, *i.e.*

$$\sum_{i=1}^{n} x_i y_i = \lambda(y) G(x) - F(y), \text{ for all } y = \alpha z, \ \alpha \geqslant 0.$$

(iii) *The function* λ *is nonnegative on the cone* K.
(iv) *The function* G *is differentiable on the open convex set* C.

From definitions 1 and 2 we see that the abstract geometric inequality is a pseudo-geometric inequality. The converse is not true, as one can see from the following theorems. The pseudo-geometric inequality represents, thus, a generalization of the abstract geometric inequality.

THEOREM 1. *Suppose that* $x = (x_1, \ldots, x_n)$ *is an arbitrary vector in* R^n *and let* $y = (y_1, \ldots, y_n)$ *be an arbitrary vector in* R^n *with non-negative components. These two vectors satisfy the inequality*

$$\sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{n} e^{x_i} + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i, \tag{2}$$

with the understanding that $y_i \ln y_i$ *is taken to be zero when* y_i *is zero.*

Moreover, this inequality becomes an equality if and only if

$$e^{x_i} = y_i, \quad i = 1, \ldots, n. \tag{3}$$

Proof. Inequality (2) can be derived in several ways. The derivation given here depends on the obvious fact that the exponential function $f : R^n \to R$ defined by

$$f(x) = \sum_{i=1}^{n} e^{x_i} \text{ for each } x = (x_1, \ldots, x_n) \text{ in } R^n,$$

is strictly convex on the R^n. Thus,

$$\sum_{i=1}^{n} e^{z_i} + \sum_{i=1}^{n} e^{z_i}(x_i - z_i) \leqslant \sum_{i=1}^{n} e^{x_i},$$

or, equivalently,

$$\sum_{i=1}^{n} e^{z_i}(1 + x_i - z_i) \leqslant \sum_{i=1}^{n} e^{x_i} \tag{4}$$

for arbitrary vectors $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_n)$ in R^n with equality holding if and only if $x = z$.

We choose an arbitrary vector $y = (y_1, \ldots, y_n)$ in R^n with positive components. Since $z = (z_1, \ldots, z_n)$ is arbitrary and y_i for $i = 1, \ldots, n$ is positive, we can choose $z_i = \ln y_i$, $i = 1, \ldots, n$. It then results from inequality (4) that

$$\sum_{i=1}^{n} y_i(1 + x_i - \ln y_i) \leqslant \sum_{i=1}^{n} e^{x_i},$$

or

$$\sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{n} e^{x_i} + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i.$$

This inequality becomes an equality if and only if

$$x_i = \ln y_i, \quad i = 1, \ldots, n$$

or

$$e^{x_i} = y_i, \quad i = 1, \ldots, n.$$

This proves theorem 1 when all components of y are positive.

If all components of y are zero, using $y_i \ln y_i = 0$ when $y_i = 0$, $i = 1, \ldots, n$, inequality (2) becomes

$$0 < \sum_{i=1}^{n} e^{x_i},$$

true, since e^{x_i} is positive for all $i = 1, \ldots, n$. The inequality (2) is a strict inequality when all components of y are zero and this proves theorem 1, because there is no vector $x = (x_1, \ldots, x_n)$ in R^n such that

$$e^{x_i} = y_i \text{ for } i = 1, \ldots, n.$$

The remaining case occurs when some of the components of y are positive and some are zero. Without loss of generality, we can assume that

$$y_i > 0 \text{ for } i = 1, \ldots, s, \tag{5}$$

$$y_i = 0 \text{ for } i = s + 1, \ldots, n, \tag{6}$$

where $1 \leqslant s < n$. From what has already been proved we know that

$$\sum_{i=1}^{s} x_i y_i \leqslant \sum_{i=1}^{s} e^{x_i} + \sum_{i=1}^{s} y_i \ln y_i - \sum_{i=1}^{s} y_i,$$

or, using (6) and $y_i \ln y_i = 0$ when $y_i = 0$,

$$\sum_{i=1}^{n} x_i y_i \leqslant \sum_{i=1}^{s} e^{x_i} + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i.$$

Since e^{x_i} is positive for $i = s + 1, \ldots, n$, we infer that

$$\sum_{i=1}^{n} x_i y_i < \sum_{i=1}^{n} e^{x_i} + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i.$$

Thus, inequality (2) is a strict inequality when some of the components of y are positive and some are zero. The proof of theorem 1 is now complete, because there is no vector $x = (x_1, \ldots, x_n)$ in R^n such that

$$e^{x_i} = y_i \text{ for } i = 1, \ldots, n.$$

THEOREM 2. *Inequality* (2) *is a pseudo-geometric inequality, but it is not an abstract geometric inequality.*

Proof. Inequality (2) is a scalar product inequality of the form (1), if in definition 1 we take $C = R^n$, the cone $K = R^n_+$ — the non-negative orthant of R^n and the functions $F, \lambda : K \to R$ and $G : C \to R$ defined by

$$F(y) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i \ln y_i, \quad y \in K,$$

$$\lambda(y) = 1, \quad y \in K,$$

$$G(x) = \sum_{i=1}^{n} e^{x_i}, \quad x \in C.$$

Evidently, in this case, the postulates (*i*), (*ii*) and (*iii*) of definition 1 are fulfilled and consequently inequality (2) is a pseudo-geometric inequality. We shall show that the postulate (*ii*) of definition 2 is not fulfilled. We shall show this by contradiction. Assume, consequently, that for each vector $x = (x_1, \ldots, x_n)$ in C, there exists a nonzero vector z in K so that

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} e^{x_i} + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i, \text{ for all } y = \alpha z, \ \alpha \geqslant 0.$$

If all components of z are positive, then for the vectors x and $y^1 = \alpha_1 z$ respectively x and $y^2 = \alpha_2 z$, where $\alpha_1 > 0$ and $\alpha_2 > 0$ with $\alpha_1 \neq \alpha_2$, inequality (2) becomes an equality. By theorem 1, the equality in (2) holds if and only if

$$e^{x_i} = y_i^1 = \alpha_1 z_i, \text{ for } i = 1, \ldots, n \tag{7}$$

respectively

$$e^{x_i} = y_i^2 = \alpha_2 z_i, \text{ for } i = 1, \ldots, n. \tag{8}$$

It then results from (7) and (8) that $\alpha_1 = \alpha_2$, contradicting the hypothesis that $\alpha_1 \neq \alpha_2$ and the theorem is proved when all components of z are positive. The remaining case occurs when some of the components of z are positive and some are zero. It then results from the proof of theorem 1 that for the vectors x and $y = \alpha z$ for all $\alpha \geqslant 0$, inequality (2) is a strict inequality, contradicting the hypothesis that inequality (2) becomes an equality for x and $y = \alpha z$ for all $\alpha \geqslant 0$. The proof of theorem 2 is now complete.

The following theorem gives a pseudo-geometric inequality which generalizes pseudo-geometric inequality (2).

THEOREM 3. *Let $x = (x_1, \ldots, x_n)$ an arbitrary vector in R^n and let $y = (y_1, \ldots, y_n)$ an arbitrary vector in R^n with non-negative components. Then*

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} e^{x_i}\right)\left(\sum_{i=1}^{n} y_i\right) + \sum_{i=1}^{n} y_i \ln y_i - \left(\sum_{i=1}^{n} y_i\right) \ln \left(\sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{n} y_i, \tag{9}$$

with the understanding that $y_i \ln y_i = 0$ if $y_i = 0$.

Moreover this inequality becomes an equality if and only if

$$e^{x_j}\left(\sum_{i=1}^{n} y_i\right) = y_j \text{ for all } j = 1, \ldots, n. \tag{10}$$

The *proof* is analogous to the proof of theorem 1. The function $f : R^n \to R$ defined by

$$f(x) = \sum_{i=1}^{n} e^{x_i} \text{ for each } x = (x_1, \ldots, x_n) \text{ in } R^n,$$

is strictly convex on R^n. Thus, (4) for arbitrary vectors $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_n)$ in R^n with equality holding if and only if $x = z$. We choose an arbitrary vector $y = (y_1, \ldots, y_n)$ in R^n with positive components. Since $z = (z_1, \ldots, z_n)$ is arbitrary and y_i for $i = 1, \ldots, n$ positive, we can choose

$$z_j = \ln \frac{y_j}{\sum_{i=1}^{n} y_i} \text{ for all } j = 1, \ldots, n.$$

It then results from inequality (4) that

$$\frac{1}{\sum_{i=1}^{n} y_i}\left[\sum_{i=1}^{n} y_i \left(1 + x_i - \ln y_i + \ln \sum_{i=1}^{n} y_i\right)\right] \leqslant \sum_{i=1}^{n} e^{x_i},$$

or, equivalently,

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} e^{x_i}\right)\left(\sum_{i=1}^{n} y_i\right) + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i \ln \left(\sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{n} y_i,$$

because $\sum_{i=1}^{n} y_i > 0$. This inequality becomes an equality if and only if

$$x_j = \ln \frac{y_j}{\sum_{i=1}^{n} y_i} \text{ for all } j = 1, \ldots, n,$$

or

$$e^{x_j}\left(\sum_{i=1}^{n} y_i\right) = y_j \text{ for all } j = 1, \ldots, n.$$

This proves theorem 3 when all components of y are positive.

If all components of y are zero, inequality (9) is satisfied, because both sides of it are zero. The remaining case occurs when some of the components of y are positive and some are zero. Without loss of generality we assume that (5) and (6) are hold. From what has already been proved, we know that

$$\sum_{i=1}^{s} x_i y_i \leqslant \left(\sum_{i=1}^{s} y_i\right)\left(\sum_{i=1}^{s} e^{x_i}\right) + \sum_{i=1}^{s} y_i \ln y_i - \sum_{i=1}^{s} y_i \ln \left(\sum_{i=1}^{s} y_i\right) - \sum_{i=1}^{s} y_i, \tag{11}$$

or, using (6) and $y_i \ln y_i = 0$ when $y_i = 0$ for $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{s} e^{x_i}\right) + \sum_{i=1}^{s} y_i \ln y_i - \sum_{i=1}^{n} y_i \ln \left(\sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{n} y_i. \tag{12}$$

Since e^{x_i} is positive and $\sum_{i=1}^{n} y_i > 0$, we infer that

$$\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{s} e^{x_i}\right) < \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} e^{x_i}\right). \tag{13}$$

From (12) and (13) we obtain

$$\sum_{i=1}^{n} x_i y_i < \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} e^{x_i}\right) + \sum_{i=1}^{n} y_i \ln y_i - \sum_{i=1}^{n} y_i \ln\left(\sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{n} y_i.$$

Thus, inequality (9) is a strict inequality when some of the components of y are positive and some are zero. The proof of theorem 1 is now complete, because there is no vector $x = (x_1, \ldots, x_n)$ in R^n such that

$$e^{x_j}\left(\sum_{i=1}^{n} y_i\right) = y_j \text{ for all } j = 1, \ldots, n.$$

THEOREM 4. *Inequality (9) is a pseudo-geometric inequality, but it is not an abstract geometric inequality.*

The *proof* is given in [2].

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