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# AN EXACT ESTIMATE IN THE THEORY OF APPROXIMATION OF THE FUNCTION $x^\alpha$ WITH BERNSTEIN POLYNOMIALS IN HAUSDORFF METRIC

V. AT. KOSTOVA
(Rousse)

**1. Introduction and main results.** This paper is a continuation of the research reported in [9]. A new estimate is obtained for the approximation of the function $f(x) = x^\alpha$, $x \in [0, 1]$, $0 < \alpha < 1$ with Bernstein polynomials in the Hausdorff metric. It is proved that this estimate is exact to the order $n^{-1}$.

We shall use the following notation:

$G^M_{[0,1]}$ — the set of all real bounded functions $g$, such that $\max\{|g(x)|,\ x \in [0, 1]\} \leq M$; $C_{[0,1]}$ — the set of continuous functions $g \in G^M_{[0,1]}$;

$$R([0, 1];\ g_1,\ g_2) = \max\{|g_1(x) - g_2(x)|,\ x \in [0, 1]\}$$

— the uniform distance between $g_1, g_2 \in G^M_{[0,1]}$;

$$\tau([0, 1];\ g_1,\ g_2) = \max\{\max_{A\in g_1}\min_{B\in g_2} \rho(A, B),\ \max_{A\in g_2}\min_{B\in g_1} \rho(A, B)\},$$

where $\rho(A, B) = \rho(A(x_1, y_1), B(x_2, y_2)) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\}$ — the Hausdorff distance between $g_1, g_2 \in C_{[0,1]}$ (For the history of Hausdorff distance, see [6]);

$$B_n(g;\ x) = \sum_{\nu=0}^{n} g\left(\frac{\nu}{n}\right)P_{n,\nu}(x), \text{ where } P_{n,\nu}(x) = \binom{n}{\nu}x^\nu(1 - x)^{n-\nu}$$

— the Bernstein polynomial for $g \in G^M_{[0,1]}$; $P_n$ — the set of all algebraic polynomials of degree $\leq n$.

The following estimates are established.

Popoviciu T. [1] and Kac M. [2] investigated the uniform approximation of $f \in \mathrm{Lip}_\alpha$ with Bernstein polynomials and proved that

$$R([0, 1];\ B_n(f),\ f) = O(n^{-\frac{\alpha}{2}}).$$

Strukov L. I. and Timan A. F. [8] investigated the uniform approximation of the function $f(x) = x^\alpha$, $0 \leq x \leq 1$, $0 < \alpha < 1$ with Bernstein polynomials and proved that

$$R([0, 1];\ B_n(f),\ x^\alpha) = O((1 - \alpha)\, n^{-\alpha}).$$

Sendov Bl. ([5]; [6], pp. 148) investigated the best Hausdorff approximation of the function $f(x) = x^\alpha$, $x \in [0, 1]$, $\alpha \in (0, 1)$ with algebraic polynomials and proved that the order of the Hausdorff approximation is independent of $\alpha$ and proved that

$$2^{6-\frac{1}{\alpha(1-\alpha)}} \cdot n^{-2} < \inf \{\tau([0, 1]; q_n, x^\alpha), q_n \in P_n\} < 2^{\frac{1}{\alpha}} \cdot n^{-2}.$$

We shall show that the Bernstein polynomials preserve this property. Thus,

THEOREM 1. *Let $f(x) = x^\alpha$, $0 \leqslant x \leqslant 1$, $0 < \alpha < 1$. Then*

$$\overline{\lim_{n\to\infty}}\ n^{-1}\tau([0, 1]; B_n(f), x^\alpha) = \frac{1-\alpha}{\alpha} O(1)$$

*holds, where*

$$B_n(f; x) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^\alpha P_{n,\nu}(x)$$

$$P_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

THEOREM 2. *Let $f(x) = x^\alpha$, $0 \leqslant x \leqslant 1$, $0 < \alpha < 1$. Then for sufficiently large $n$ it holds that*

$$(a) \qquad \tau([0, 1]; B_n(f), x^\alpha) \leqslant 16 \cdot \frac{1-\alpha}{\alpha} \cdot \frac{1}{n};$$

$$(b) \qquad \tau([0, 1]; B_n(f), x^\alpha) \geqslant \text{const} \cdot \frac{1-\alpha}{\alpha} \cdot \frac{1}{n}.$$

**2. Proofs of the theorems.** First, we shall note that Theorem 1 is an immediate consequence from Theorem 2. The heart of the proof of Theorem 2 is the following.

LEMMA. *Let $f(x) = x^\alpha$, $0 \leqslant x \leqslant 1$, $0 < \alpha < 1$. Then for $x \in (16(1-\alpha)n^{-1}, 1]$ it holds that*

$$(1) \qquad x^\alpha - B_n(f; x) \leqslant 16(1-\alpha)x^{\alpha-1} \cdot n^{-1}.$$

*Proof.* This statement is proven in three parts.

1) Let $x \in \left[16(1-\alpha)k^{-1}\ln k, \frac{1}{2}\right]$, $k > n$.

Our purpose is to find a lower estimate for $B_k(t^\alpha; x) - B_{k-1}(t^\alpha; x)$. Using the definition of Bernstein polynomials, after some elementary calculation, we obtain

$$(2) \qquad B_k(t^\alpha; x) - B_{k-1}(t^\alpha; x)$$

$$= \sum_{\nu=1}^{k-1} \frac{\nu_\alpha}{k(k-1)^\alpha}\left[\varphi\left(\frac{1}{\nu}\right) - \varphi\left(\frac{1}{k}\right)\right] P_{k,\nu}(x),$$

where $\varphi(x) = [1-(1-x)^\alpha]/x \cdot$ [8].



Now, we shall prove that for $\nu = 1, 2, \ldots, k-1$ it holds that

$$(3) \qquad \nu\left[\varphi\left(\frac{1}{\nu}\right) - \varphi\left(\frac{1}{k}\right)\right] \leqslant (1-\alpha).$$

We use the inequality

$$1-(1-x)^\alpha \leqslant \alpha x + (1-\alpha)x^2$$

and in view of the definition of $\varphi$, we get

$$\varphi(x) < \alpha + (1-\alpha)x$$

or

$$(4) \qquad 1-\varphi(x) > (1-\alpha)(1-x).$$

From the definition of $\varphi$, it follows that $\varphi$ is increasing on $[0, 1]$. Therefore, the function $1-\varphi(t)$ is decreasing on $[0, 1]$ and $1-\varphi(t) \leqslant 1-\alpha$. Using this, after some elementary calculations, we get

$$1-\varphi(x) > (1-\varphi(t))(1-x)$$

or

$$(5) \qquad \varphi(x) - \varphi(t) \leqslant x(1-\alpha).$$

It is evident that (5) yields (3), if we set $x = 1/\nu$, $t = 1/k$.

Further, we express

$$(6) \qquad B_k(t^\alpha; x) - B_{k-1}(t^\alpha; x) =$$

$$= \frac{1}{k(k-1)^\alpha} \cdot \sum_{\left|x-\frac{\nu}{k}\right| \leqslant 2\delta(x;k)} \nu^\alpha\left[\varphi\left(\frac{1}{\nu}\right) - \varphi\left(\frac{1}{k}\right)\right] P_{k,\nu}(x) +$$

$$+ \frac{1}{k(k-1)^\alpha} \cdot \sum_{\left|x-\frac{\nu}{k}\right| > 2\delta(x;k)} \nu^\alpha\left[\varphi\left(\frac{1}{\nu}\right) - \varphi\left(\frac{1}{k}\right)\right] P_{k,\nu}(x),$$

where $\delta(x; k) = [(1-\alpha)x(1-x)k^{-1}\ln k]^{1/2}$.

In view of (3) for the first sum, we obtain

$$\frac{1}{k(k-1)} \sum_{\left|x-\frac{\nu}{k}\right| \leqslant 2\delta(x;k)} \left(\frac{\nu}{k-1}\right)^{\alpha-1} \left\{\nu\left[\varphi\left(\frac{1}{\nu}\right) - \varphi\left(\frac{1}{k}\right)\right]\right\} P_{k,\nu}(x) \leqslant$$

$$\leqslant \frac{(1-\alpha)2^{\alpha-1}}{k(k-1)} \sum_{\left|x-\frac{\nu}{k}\right| \leqslant 2\delta(x;k)} \left(\frac{\nu}{k}\right)^{\alpha-1} P_{k,\nu}(x) \leqslant \frac{(1-\alpha)2^{\alpha-1}}{k(k-1)} [x - 2\delta(x;k)]^{\alpha-1} \leqslant$$

$$\leqslant \frac{(1-\alpha)\cdot 2^{\alpha-1}\cdot x^{\alpha-1}}{k(k-1)} \left\{1 - 2\left[(1-\alpha)\cdot\frac{1-x}{x}\cdot k^{-1}\ln k\right]^{1/2}\right\}^{\alpha-1} \leqslant$$

$$\leqslant \frac{1-\alpha}{k(k-1)} \cdot x^{\alpha-1}.$$

Further we need the following proposition, which was proven in [4]
If $0 \leqslant x \leqslant 1$ and $0 \leqslant z \leqslant 3/2[kx(1-x)]^{1/2}$, then

$$\sum_{\left|x-\frac{\nu}{k}\right|>2\sigma} P_{k,\nu}(x) \leqslant 2\exp(-z^2), \tag{7}$$

where $\sigma = z[k^{-1}x(1-x)]^{1/2}$.

Now, using (3) and (7) for the second sum, we have

$$\frac{1}{k(k-1)^{\alpha}} \cdot \sum_{\left|x-\frac{\nu}{k}\right|>2\delta(x;\,k)} \nu^{\alpha}\left[\varphi\left(\frac{1}{\nu}\right)-\varphi\left(\frac{1}{k}\right)\right]P_{k,\nu}(x) \leqslant$$

$$\leqslant \frac{1-\alpha}{k(k-1)^{\alpha}} \sum_{\left|x-\frac{\nu}{k}\right|>2\delta(x;\,k)} P_{k,\nu}(x) \leqslant \frac{2(1-\alpha)}{k(k-1)^{\alpha}} \cdot \frac{1}{k^{1-\alpha}} \leqslant \frac{2(1-\alpha)}{k(k-1)}.$$

Hence, for $x \in [16(1-\alpha)k^{-1}\ln k,\ 1/2]$, it holds that

$$B_k(t^{\alpha};\ x) - B_{k-1}(t^{\alpha};\ x) \leqslant \frac{(1-\alpha)}{k(k-1)} \cdot x^{\alpha-1} + \frac{2(1-\alpha)}{k(k-1)}.$$

It is known that the sequence of Bernstein polynomials $\{B_n(f)\}_{n=1}^{\infty}$ for the function $f(x) = x^{\alpha}$ converges to $f$.
Therefore,

$$x^{\alpha} - B_n(t^{\alpha};\ x) = \sum_{k=n+1}^{\infty}[B_k(t^{\alpha};\ x) - B_{k-1}(t^{\alpha};\ x)] \tag{8}$$

$$\leqslant \sum_{k=n+1}^{\infty}\left[\frac{1-\alpha}{k(k-1)} \cdot x^{\alpha-1} + \frac{2(1-\alpha)}{k(k-1)}\right] \leqslant$$

$$\leqslant \frac{1-\alpha}{n}\,x^{\alpha-1} + \frac{2(1-\alpha)}{n} \leqslant \frac{2(1-\alpha)}{n} \cdot x^{\alpha-1}.$$

2) Let $x \in (16(1-\alpha)k^{-1},\ 16(1-\alpha)k^{-1}\ln k)$, $k > n$.
Without any restrictions, we can assume that there exists
$0 < \gamma_x < 1$ such that

$$x \in [16(1-\alpha)k^{-1}(\ln k)^{1-\gamma_x},\ 16(1-\alpha)k^{-1}\ln k).$$

We define

$$\delta_{\gamma_x}(x;\ k) = [(1-\alpha)\,x(1-x)k^{-1}(\ln k)^{1-\gamma_x}]^{1/2}$$

and express $B_k(f;\ x) - B_{k-1}(f;\ x)$ as in (6).

For the first sum, we have

$$\frac{1}{k(k-1)} \sum_{\left|x-\frac{\nu}{k}\right|\leqslant 2\delta_{\gamma_x}(x;\,k)} \left(\frac{\nu}{k-1}\right)^{\alpha-1}\left\{\nu\left[\varphi\left(\frac{1}{\nu}\right)-\varphi\left(\frac{1}{k}\right)\right]\right\}P_{k,\nu}(x) \leqslant \tag{9}$$



$$\geqslant \frac{(1-\alpha)\cdot 2^{\alpha-1}}{k(k-1)} \cdot \sum_{\left|x-\frac{\nu}{k}\right|\leqslant 2\delta_{\gamma_x}(x;\,k)} \left(\frac{\nu}{k}\right)^{\alpha-1} P_{k,\nu}(x) \leqslant$$

$$\leqslant \frac{(1-\alpha)\cdot 2^{\alpha-1}}{k(k-1)} \cdot x^{\alpha-1} \cdot \left\{1 - 2\left[(1-\alpha)\cdot\frac{1-x}{x}\cdot k^{-1}(\ln k)^{1-\gamma_x}\right]^{1/2}\right\}^{\alpha-1} \leqslant$$

$$\leqslant \frac{1-\alpha}{k(k-1)} \cdot x^{\alpha-1}$$

In view of (3) and (7) for the second sum, we get

$$\frac{1}{k(k-1)^{\alpha}} \cdot \sum_{\left|x-\frac{\nu}{k}\right|>2\delta_{\gamma_x}(x;\,k)} \nu^{\alpha}\left[\varphi\left(\frac{1}{\nu}\right)-\varphi\left(\frac{1}{k}\right)\right]P_{k,\nu}(x) \leqslant \tag{10}$$

$$\leqslant \frac{1-\alpha}{k(k-1)^{\alpha}} \sum_{\left|x-\frac{\nu}{k}\right|>2\delta_{\gamma_x}(x;\,k)} \cdot P_{k,\nu}(x) \leqslant$$

$$\leqslant \frac{2(1-\alpha)}{k(k-1)^{\alpha}} \exp[-(1-\alpha)\cdot(\ln k)^{1-\gamma_x}].$$

Now we shall prove that for

$$x \in [16(1-\alpha)k^{-1}(\ln k)^{1-\gamma_x},\ 16(1-\alpha)k^{-1}\ln k),\ 0 < \gamma_x < 1$$

the inequality

$$\exp[-(1-\alpha)(\ln k)^{1-\gamma_x}] \leqslant 16^{1-\alpha}\cdot k^{\alpha-1}\cdot x^{\alpha-1} \tag{11}$$

holds, if $k$ is sufficiently large. For this purpose it is sufficient to show that the statement is true for $x = 16(1-\alpha)k^{-1}\ln k$, i.e. it is necessary to prove that

$$\exp[-(1-\alpha)(\ln k)^{1-\gamma_x}] \leqslant [(1-\alpha)\ln k]^{-1}$$

holds. But this inequality is valid as

$$\ln[(1-\alpha)\ln k] \leqslant (\ln k)^{1-\gamma_x}$$

for sufficiently large $k$.

Therefore, (10) and (11) yield

$$\frac{2(1-\alpha)}{k(k-1)^{\alpha}}\exp[-(1-\alpha)\cdot(\ln k)^{1-\gamma_x}] \leqslant \frac{10(1-\alpha)}{k(k-1)} \cdot x^{\alpha-1}. \tag{12}$$

Then from (9) and (12), we get

$$B_k(t^{\alpha};\ x) - B_{k-1}(t^{\alpha};\ x) \leqslant \frac{16(1-\alpha)}{k(k-1)} \cdot x^{\alpha-1}.$$

Further, using the method described above, we can conclude that for

$$x \in (16(1-\alpha)k^{-1};\ 16(1-\alpha)k^{-1} \cdot \ln k)$$

it holds that

$$(13) \qquad x^{\alpha} - B_n(t^{\alpha};\ x) \leqslant \frac{16(1-\alpha)}{n} \cdot x^{\alpha-1}.$$

3) Let $x \in (1/2,\ 1]$.
In view of the theorem of Woronovsca ([3], pp. 246), we get

$$(14) \qquad x^{\alpha} - B_n(t^{\alpha};\ x) \leqslant \frac{\alpha(1-\alpha)}{n} \cdot x^{\alpha-1}.$$

Inequalities (8), (13) and (14) yield (1). The Lemma is proven.
Now we are ready to prove Theorem 2.
First, we shall prove that, for sufficiently large $n$,

$$(15) \qquad \tau(\Delta_1;\ B_n(t^{\alpha}),\ x^{\alpha}) \leqslant \frac{16(1-\alpha)}{\alpha n},$$

when $\Delta_1 = [16(1-\alpha)n^{-1}\ln n,\ 1]$. This statement is true as $f(x) = x^{\alpha}$ is a convex function and in view of (8) it follows that

$$x^{\alpha} - \frac{1-\alpha}{n} x^{\alpha-1} \cdot \frac{2(1-\alpha)}{n} \leqslant B_n(t^{\alpha};\ x) \leqslant x^{\alpha}$$

or

$$\left(x - \frac{16(1-\alpha)}{\alpha} \cdot \frac{1}{n}\right)^{\alpha} \leqslant B_n(t^{\alpha};\ x) \leqslant x^{\alpha},$$

which, according to the definition of Hausdorff distance, implies (15).
Let now $x_0 \in \Delta_2 = (16(1-\alpha) \cdot n^{-1};\ 16(1-\alpha)n^{-1}\ln n]$ and the Hausdorff distance in this point is $\delta_0$. Then from the definition of the Hausdorff distance, we obtain

$$B_n(t^{\alpha};\ x_0) = (x_0 - \delta_0)^{\alpha} - \delta_0 = x_0^{\alpha} - \text{Const}_1 \cdot \alpha \cdot \delta_0 \cdot x_0^{\alpha-1} - \delta_0.$$

Hence, Hausdorff distance in the point $x_0$ can be estimated with uniform as follows

$$(16) \qquad \text{Const}_1 \cdot \alpha \cdot \delta_0 \cdot x_0^{\alpha-1} + \delta_0 = x_0^{\alpha} - B_n(t^{\alpha};\ x_0),\ /1 < \text{Const}_1 < 2/.$$

Then Lemma implies that

$$\alpha\delta_0 x_0^{\alpha-1} + \delta_0 \leqslant \frac{16(1-\alpha)}{n} \cdot x_0^{\alpha-1}.$$

Therefore, for $x \in \Delta_2$ and sufficiently large $n$, the Hausdorff distance $\delta$ satisfies the condition

$$\delta \leqslant \frac{16(1-\alpha)}{\alpha} \cdot \frac{1}{n}.$$

The first part $(a)$ of Theorem 2 is proved.



For the proof of the second part $(b)$ we use the inequality

$$B_k(t^{\alpha};\ x) - B_{k-1}(t^{\alpha};\ x) > (k-1)^{-\alpha}\, x(1-x)^{k-1}\left[1 - \varphi\left(\frac{1}{k}\right)\right],$$

which follows directly from (3). Then

$$(17) \qquad x^{\alpha} - B_n(t^{\alpha};\ x) = \sum_{k=n+1}^{\infty} [B_k(t^{\alpha};\ x) - B_{k-1}(t^{\alpha};\ x)] \geqslant$$

$$\geqslant x \sum_{k=n+1}^{\infty} (k-1)^{-\alpha}(1-x)^{k-1}\left[1 - \varphi\left(\frac{1}{k}\right)\right] \geqslant$$

$$\geqslant x(2n)^{-\alpha}\left[1 - \varphi\left(\frac{1}{n+1}\right)\right] \sum_{k=n+1}^{\infty} (1-x)^{k-1}$$

$$\geqslant (2n)^{-\alpha}\left[1 - \varphi\left(\frac{1}{n+1}\right)\right][(1-x)^{n} - (1-x)^{2n}].$$

We set $x^* = 16(1-\alpha)n^{-1}$, $\delta^*$ – the Hausdorff distance in the point $x^*$ and according to (15), we obtain

$$(x^*)^{\alpha} - B_n(t^{\alpha};\ x^*) \geqslant (2n)^{-\alpha}\left[1 - \varphi\left(\frac{1}{n+1}\right)\right]\left[\left(1 - \frac{16(1-\alpha)}{n}\right)^{n} - \left(1 - \frac{16(1-\alpha)}{n}\right)^{2n}\right].$$

But

$$\lim_{n\to\infty}\left[1 - \varphi\left(\frac{1}{n+1}\right)\right]\cdot\left[\left(1 - \frac{16(1-\alpha)}{n}\right)^{n} - \left(1 - \frac{16(1-\alpha)}{n}\right)^{2n}\right] \geqslant \frac{1-\alpha}{2e^{16(1-\alpha)}}.$$

Hence, for sufficiently large $n$, it is true that

$$2\alpha\delta^*(x^*)^{\alpha-1} + \delta^* \geqslant (x^*)^{\alpha} - B_n(t^{\alpha};\ x^*) \geqslant \frac{1-\alpha}{4e^{16(1-\alpha)}} \cdot \frac{1}{n^{\alpha}}$$

or

$$\delta^* \geqslant \frac{1}{4e^{16}} \cdot \frac{1-\alpha}{\alpha} \cdot \frac{1}{n}.$$

Therefore there exists a point of the interval $[0,\ 1]$, such that

$$\tau([0,\ 1];\ B_n(t)^{\alpha};\ x^{\alpha}) \geqslant \text{const.}\ \frac{1-\alpha}{\alpha} \cdot \frac{1}{n}.$$

This completes the proof of Theorem 2.

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*Department of Mathematics,*
*TU "Angel Kunchev"*
*7004 Rousse*
*Bulgaria*