MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 16, N° 2, 1987, pp. 141—147

ON A RUNGE-KUTTA TYPE METHOD

MARIAN MUREȘAN
(Cluj-Napoca)

0. Introduction. There is much literature concerning the explicit Runge-Kutta methods (see, e.g., [3]). These methods are, by their nature, one-step methods. An interesting idea is contained in [2], where a suitable change is performed by which at each step we take into account some results of the previous step, this new method of integration being a two-step method. The advantage lies in the fact that at each step we have to do $n-1$ evaluations of the function for the n^{th} order method, $n \in \{3, 4\}$. This cost reduction is important for non-trivial equations.

It is the purpose of the present paper to extend the results from [2] for the vector case.

We shall use the notation introduced in [3], pp. 119—120, and 132 for A, B, C, D, E, F, G, H, I, J, K, L, M, N, P, Q, R.

We consider a system of differential equations of the first order:

$$(1)\qquad \begin{aligned} \frac{dy_1}{dx} &= f_1(y_1, y_2, \ldots, y_s) \\ \frac{dy_2}{dx} &= f_2(y_1, y_2, \ldots, y_s) \\ &\vdots \\ \frac{dy_s}{dx} &= f_s(y_1, y_2, \ldots, y_s). \end{aligned}$$

Without loss of generality, throughout this paper, only autonomous systems of differential equations will be considered. As is shown in [1], system (1) may by written in a vector form:

$$(2)\qquad \frac{d\mathbf{y}}{dx} = \mathbf{f}(\mathbf{y}).$$

We suppose that the values $\mathbf{y} = \mathbf{y}_0$ are given at $x = a$. The interval $[a, b]$, where the independent variable x runs, is partitioned: $h_i > 0$ and:

$$a + \sum_{i=1}^{m} h_i = b.$$

As usually, we suppose the function $\mathbf{f}$ to be sufficiently smooth when the variable x belongs to an interval which contains the given interval $[a, b]$ such that all that follows is correct. This implies, among others, that the Cauchy problem (2) + ($\mathbf{y} = \mathbf{y}_0$) has a unique solution which, we assume, is defined on the interval $[a, b]$.

1. The 3^{rd}-order method. The basic relations are :

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h_n(\alpha_0\mathbf{K}_{0,n} + \alpha_1\mathbf{K}_{1,n}), \tag{1.1}$$

where :

$$\mathbf{K}_{i,n} = \mathbf{f}(\mathbf{Arg}_{i,n}),\ i \in \{0, 1\},$$
$$\mathbf{Arg}_{0,n} = \mathbf{y}_n + h_n(\lambda_{00}\mathbf{K}_{0,n-1} + \lambda_{01}\mathbf{K}_{1,n-1}),$$
$$\mathbf{Arg}_{1,n} = \mathbf{y}_n + h_n(\lambda_{10}\mathbf{K}_{0,n-1} + \lambda_{11}\mathbf{K}_{1,n-1} + \rho\mathbf{K}_{0,n}),$$
$$n = 0, 1, \ldots, m.$$

At the first step, we take : $\mathbf{K}_{0,-1} = \mathbf{K}_{1,-1} = \mathbf{y}_0$. Let us denote :

$$\mu_0 = \lambda_{00} + \lambda_{01},\ \mu_1 = \lambda_{10} + \lambda_{11} + \rho. \tag{1.2}$$

To get the third-order accuracy, we write (1.1) in the following form :

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \sum_{i=1}^{4}\frac{1}{i\,!}\left(\sum_{j=1}^{i}\binom{i}{j}h_{n-1}^{i-j}h_n^j\mathbf{c}_{i-j,j}\right) \tag{1.3}$$

and try to identify (1.3) with the expansion of the function $\mathbf{y}$, similar to (1.3). The following relations result :

$$\alpha_0 + \alpha_1 = 1, \tag{1.4}$$

$$2(\alpha_0\mu_0 + \alpha_1\mu_1) = 1, \tag{1.5}$$

$$3(\alpha_0\mu_0^2 + \alpha_1\mu_1^2) = 1, \tag{1.6}$$

$$6\alpha_1\rho\mu_0 = 1, \tag{1.7}$$

$$2(\alpha_0(\lambda_{00}\mu_0 + \lambda_{01}\mu_1) + \alpha_1(\lambda_{10}\mu_0 + \lambda_{11}\mu_1 + \rho)) = 1. \tag{1.8}$$

This last relation may be written in an equivalent form if we use (1.5)

$$(\mu_0 - 1)(\alpha_0\lambda_{00} + \alpha_1\lambda_{10} + (\mu_1 - 1)(\alpha_0\lambda_{01} + \alpha_1\lambda_{11}) = 0. \tag{1.9}$$

From (1.4), (1.5) and (1.6), it results that $\mu_0 \neq \mu_1$ and

$$\alpha_0 = \frac{2\mu_1 - 1}{2(\mu_1 - \mu_0)}, \tag{1.10}$$

$$\alpha_1 = \frac{2\mu_0 - 1}{2(\mu_0 - \mu_1)}, \tag{1.11}$$

and the compatibility condition

$$6\mu_0\mu_1 - 3(\mu_0 + \mu_1) + 2 = 0. \tag{1.12}$$

We have :

$$\rho = \frac{(\mu_1 - \mu_0)(2\mu_1 - 1)}{\mu_0}. \tag{1.13}$$

From (1.2) and (1.8) it results, like in [2], that

$$\lambda_{0i} = (-1)^i\left(\frac{1 - \mu_{1-i}}{\mu_0 - \mu_1}\mu_0 + 2\alpha_1\gamma\right), \tag{1.14}$$

$$\lambda_{1i} = (-1)^i\left(\frac{(\mu_1 - \rho)(1 - \mu_{1-i})}{\mu_0 - \mu_1} - 2\alpha_0\gamma\right), \tag{1.15}$$

where $i \in \{0, 1\}$ and γ is a real parameter.

The truncation error is

$$R_n = \delta_1 h_n^4 + \delta_2 h_n^2 h_{n-1}^2 + \gamma R_n^* + O(h^5),$$

where

$$\delta_1 = \frac{1 - 6\mu_0\mu_1}{216}\mathbf{E} + \frac{3 - 4\mu_1}{24}\mathbf{F} + \frac{1 - 2\mu_0}{24}\mathbf{G} + \frac{1}{24}\mathbf{H},$$

$$\delta_2 = \frac{-1 + \mu_1}{12\mu_0}((3\mu_0 - 1)(\mu_0 - 1)\,\mathbf{G} + (1 - 4\mu_0)\,\mathbf{H}),$$

$$R_n^* = -\frac{h_{n-1}h_n^3}{6}\left(\mathbf{F} + \frac{1}{\mu_0}\mathbf{H}\right).$$

Several special cases are considered in [2] when $s = 2$ and

$$\frac{dy_1}{dx} = 1.$$

Remark. We note that if $\gamma = 0$, then the following relations hold :

$$\lambda_{00}\mu_0 + \lambda_{01}\mu_1 = \mu_0, \tag{1.16}$$

$$\lambda_{10}\mu_0 + \lambda_{11}\mu_1 + \rho = \mu_1. \tag{1.17}$$

These relations have close analogous with the case of the 4^{th}-order method ((2.13), (2.14)).

2. The 4^{th}-order method. The similars of (1.1) are

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h_n(\alpha_0\mathbf{K}_{0,n} + \alpha_1\mathbf{K}_{1,n} + \alpha_2\mathbf{K}_{2,n}), \tag{2.1}$$

$$\mathbf{K}_{i,n} = \mathbf{f}(\mathbf{Arg}_{i,n}),\ i = 0, 1, 2,$$

$$\mathbf{Arg}_{0,n} = \mathbf{y}_n + h_n \left(\sum_{i=0}^{2} \lambda_{0i} \mathbf{K}_{i,n-1} \right),$$

$$\mathbf{Arg}_{1,n} = \mathbf{y}_n + h_n \left(\sum_{i=0}^{2} \lambda_{1i} \mathbf{K}_{i,n-1} + \rho_{10} \mathbf{K}_{0,n} \right),$$

$$\mathbf{Arg}_{2,n} = \mathbf{y}_n + h_n \left(\sum_{i=0}^{2} \lambda_{2i} \mathbf{K}_{i,n-1} + \rho_{20} \mathbf{K}_{0,n} + \rho_{21} \mathbf{K}_{1,n} \right),$$

$n = 0, 1, \ldots, m.$

For the first step, we take

$$\mathbf{K}_{i,-1} = \mathbf{y}_0, \quad i = 0, 1, 2.$$

We write (2.1) as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \sum_{i=1}^{5} \frac{1}{i!} \left(\sum_{j=1}^{i} \binom{i}{j} h_{n-1}^{i-j} h_n^j \mathbf{c}_{i-j,j} \right) \tag{2.2}$$

and use an analogous of (11) from [1] to get accurate up to the terms in h^4. Then the following relations concerning the parameters of the method have to hold:

$$\lambda_{00} + \lambda_{01} + \lambda_{02} = \mu_0, \tag{2.3}$$

$$\lambda_{10} + \lambda_{11} + \lambda_{12} + \rho_{10} = \mu_1, \tag{2.4}$$

$$\lambda_{20} + \lambda_{21} + \lambda_{22} + \rho_{20} + \rho_{21} = \mu_2, \tag{2.5}$$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1, \tag{2.6}$$

$$2(\alpha_0\mu_0 + \alpha_1\mu_1 + \alpha_2\mu_2) = 1, \tag{2.7}$$

$$3(\alpha_0\mu_0^2 + \alpha_1\mu_1^2 + \alpha_2\mu_2^2) = 1, \tag{2.8}$$

$$4(\alpha_0\mu_0^3 + \alpha_1\mu_1^3 + \alpha_2\mu_2^3) = 1, \tag{2.9}$$

$$6(\alpha_1\rho_{10}\mu_0 + \alpha_2\rho_{20}\mu_0 + \alpha_2\rho_{21}\mu_1) = 1, \tag{2.10}$$

$$8(\alpha_1\rho_{10}\mu_0\mu_1 + \alpha_2\mu_2(\rho_{20}\mu_0 + \rho_{21}\mu_1)) = 1, \tag{2.11}$$

$$12(\alpha_1\rho_{10}\mu_0^2 + \alpha_2(\rho_{20}\mu_0^2 + \rho_{21}\mu_1^2)) = 1, \tag{2.12}$$

$$24\,\alpha_2\rho_{10}\rho_{21}\mu_0 = 1, \tag{2.13}$$

$$\lambda_{00}\mu_0 + \lambda_{01}\rho_1 + \lambda_{02}\rho_2 = \mu_0, \tag{2.14}$$

$$\lambda_{10}\mu_0 + \lambda_{11}\mu_1 + \lambda_{12}\mu_2 + \rho_{10} = \mu_1, \tag{2.15}$$

$$\lambda_{20}\mu_0 + \lambda_{21}\mu_1 + \lambda_{22}\mu_2 + \rho_{20} + \rho_{21} = \mu_2, \tag{2.16}$$

$$2\left(\sum_{i=0}^{2} \alpha_i \sum_{k=0}^{2} \lambda_{ik}\mu_k + S \right) = 1, \tag{2.17}$$

$$2\left(\sum_{i=1}^{2} \alpha_i \sum_{k=0}^{2} \lambda_{ik}\mu_k^2 + S \right) = 1, \tag{2.18}$$

$$2\left(\sum_{i=0}^{2} \alpha_i(\lambda_{i1}\rho_{10}\mu_0 + \lambda_{i2}(\rho_{20}\mu_0 + \rho_{21}\mu_1)) + S \right) = 1, \tag{2.19}$$

where $S = \alpha_1\rho_{10} + \alpha_2(\rho_{20} + \rho_{21})$.

We shall express these parameters in terms of μ_1, μ_2, μ_0.

From (2.6)–(2.9) the following compatibility relation results

$$12\mu_0\mu_1\mu_2 - 6(\mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2) + 4(\mu_0 + \mu_1 + \mu_2) - 3 = 0. \tag{2.20}$$

From (2.10)–(2.13) another compatibility relation results

$$12\mu_0\mu_1\mu_2 - 6\mu_0\mu_1 - 6\mu_1\mu_2 - 8\mu_0\mu_2 + 6\mu_0 + 4\mu_1 + 4\mu_2 - 3 = 0. \tag{2.21}$$

The last two equalities provide:

$$\mu_0(\mu_2 - 1) = 0. \tag{2.22}$$

But, if $\mu_0 = 0$, then (2.13) fails, so with necessity:

$$\mu_2 = 1. \tag{2.23}$$

Now, (2.6)–(2.8) provide:

$$\alpha_0 = \frac{3\mu_1 - 1}{6(1 - \mu_0)(\mu_1 - \mu_0)}, \tag{2.24}$$

$$\alpha_1 = \frac{3\mu_0 - 1}{6(\mu_1 - 1)(\mu_1 - \mu_0)}, \tag{2.25}$$

$$\alpha_2 = \frac{1 - 6\mu_0\mu_1}{12(1 - \mu_1)(1 - \mu_0)}, \tag{2.26}$$

and equations (2.10)–(2.12) give

$$\rho_{10} = \frac{\mu_1 - \mu_0}{4\mu_0(1 - 3\mu_0)}, \tag{2.27}$$

$$\rho_{20} = \frac{(1 - \mu_0)(4\mu_1 - 1)(3\mu_0 - \mu_1)}{2\mu_0(\mu_1 - \mu_0)(1 - 6\mu_0\mu_1)}, \tag{2.28}$$

$$\rho_{21} = \frac{(1 - 3\mu_0)(1 - 4\mu_0\mu_1)}{(\mu_1 - \mu_0)(1 - 6\mu_0\mu_1)}. \tag{2.29}$$

In the sequel, we show that $S = 1/2$. It results from (2.10) and from the identity:

$$\alpha_2\rho_{21}(\mu_0 - \mu_1) = \frac{2\mu_0 - 1}{12\mu_1}.$$

Substituting $S = 1/2$ in (2.7), (2.17) and (2.18), we get a Vandermonde system

$$\sum_{k=0}^{2} \sum_{i=0}^{2} \alpha_i \lambda_{ik} = 0,$$

$$\sum_{k=0}^{2} \mu_k \sum_{i=0}^{2} \alpha_i \lambda_{ik} = 0,$$

$$\sum_{k=0}^{2} \mu_k^2 \sum_{i=0}^{2} \alpha_i \lambda_{ik} = 0,$$

from where it results that

$$\sum_{i=0}^{2} \alpha_i \lambda_{ik} = 0, \quad k = 0, 1, 2. \tag{2.30}$$

Remark. We note that (2.19) follows from (2.30) and $S = 1/2$.

From (2.3)–(2.5) and (2.14)–(2.16) it follows that

$$\lambda_{i0}(\mu_0 - 1) + \lambda_{i1}(\mu_1 - 1) = 0, \quad i = 0, 1, 2. \tag{2.31}$$

Now, taking into account (2.30), (2.31) and (2.3)–(2.5), we get a linear system with 9 unknows λ_{ik} $i, k \in \{0, 1, 2\}$. The rank of this system is equal to 7. We take $\lambda_{11} = \beta$, $\lambda_{21} = \gamma$; then

$$\lambda_{10} = -\frac{\mu_1 - 1}{\mu_0 - 1}\beta, \quad \lambda_{20} = -\frac{\mu_1 - 1}{\mu_0 - 1}\gamma, \quad \lambda_{01} = -\frac{1}{\alpha_0}(\alpha_1\beta + \alpha_2\gamma),$$

$$\lambda_{00} = -\frac{\mu_1 - 1}{\mu_0 - 1} \cdot \frac{\alpha_1\beta + \alpha_2\gamma}{\alpha_0}, \quad \lambda_{02} = \mu_0 - \frac{\mu_1 - \mu_0}{\alpha_0(\mu_0 - 1)}(\alpha_1\beta + \alpha_2\gamma),$$

$$\lambda_{12} = \mu_1 - \rho_{10} + \frac{\mu_1 - \mu_0}{\mu_0 - 1} \cdot \beta,$$

$$\lambda_{22} = \mu_2 - (\rho_{20} + \rho_{21}) + \frac{\mu_1 - \mu_0}{\mu_0 - 1} \cdot \gamma.$$

We have found the following truncation error:

$$R_n = \delta_1 h_n^5 + \delta_2 h_n^3 h_{n-1}^2 + O(h^6),$$

where

$$\delta_1 = \frac{10\mu_0\mu_1 - 1}{2880}(\mathbf{I} - 4\mathbf{M}) + \frac{5\mu_1 - 3}{240}(\mathbf{J} - 2\mathbf{N}) +$$

$$+ \frac{5\mu_0 - 2}{240}(\mathbf{K} - \mathbf{P}) + \frac{1}{120}(\mathbf{Q} + 4\mathbf{L}) +$$

$$\frac{(6\mu_0\mu_1 - 1)(12 - 5\mu_1 - 43\mu_0) + 10(1 - \mu_0)(1 - 5\mu_0)(3 - 4\mu_1)}{480(1 - 4\mu_0)(6\mu_0\mu_1 - 1)} \mathbf{R},$$

and δ_2 is a sum of the form:

$$\delta_2 = c_1\mathbf{K} + c_2\mathbf{L} + c_3\mathbf{P} + c_4\mathbf{Q},$$

c_i $i = \overline{1, 4}$ being the coefficients depending on the parameters μ_0 and μ_1.

Some considerations on the stability of the method and several comparative examples will be published in a forthcoming paper.

REFERENCES

[1] Butcher, J. C., *Coefficients for the study of Runge-Kutta integration processes*, J. Austral. Math. Soc., 3 (1963), 185–201.

[2] Gross, W., Urbani, A. M., *Economized Runge-Kutta methods*, Rend. di Mat. e delle sue appl., s. VII, 2 (1982), 227–235.

[3] Henrici, P., *Discrete variable methods in ordinary differential equations*, Wiley, New York, London, 1962.

Received 26.III.1987

Institutul Politehnic
Catedra de matematică
Str. Emil Isac, nr. 5
3400 Cluj-Napoca
România