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CERTAIN EXTENDED RULES FOR NUMERICAL INTEGRATION

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Abstract. Certain Kronrod-type rules with their error estimates have more simply been derived by the use of interpolation coefficients in terms of the Fourier coefficients. We then obtain explicitly a family of extended rules for $\int_{C_1} F(Z)\,|dZ|$, $|dZ| = ds$, $C_1 : |Z| = 1$, where $F(Z) = f\left(\frac{Z+Z^{-1}}{2}\right)$ and their error bounds found by use of Laurants expansion. Meanwhile, we show that certain Kronrod rules give rise to the generalized Gauss integration rules.

1. Introduction. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$ are those polynomials which are orthogonal with respect to the weight function $W^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ and h_n is the normalizing factor given by $h_n\delta_{mn} = \int_{-1}^{1} W^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x$. It is well known that the rules which have the maximum degree of exactness (or polynomial degree) are the so-called Gauss-Jacobi integration rules (GJIR) of the type :

$$I(f) = \int_{-1}^{1} W^{(\alpha,\beta)}(x) f(x)\mathrm{d}x = \sum_{i=1}^{n} H_{n,i} f(\zeta_{n,i}) + R_n(f), \tag{1}$$

with $R_n(f) = 0$, whenever $f(x)$ is a polynomial of degree $2n-1$. The Kronrod extension of GJIR is given by

$$I(f) = \sum_{i=1}^{n} u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + E_n(f), \tag{2}$$

with the degree of exactness $3n+1$, where $\{x_i\}_1^n$ are the zeroes of polynomials orthogonal on $[a, b]$ with respect to $W^{(\alpha,\beta)}(x)$ and y_i's are the zeroes of certain polynomial $E_{n+1,n}(x)$.

The first to discover (2) was Kronrod who dealt with the case $(\alpha, \beta) = (0, 0)$, the Gauss-Legendre rule. Subsequently, Patterson (1967), Piesses & Braders (1974) and Monegato (1976) improved on Kronrod's original work. Monegato (1976, 79) points out the Kronrod extension of

n-point GJIR corresponding to $(\alpha, \beta) = (-1/2, -1/2)$ and $(1/2, 1/2)$ in explicit forms. The rules are exact for polynomials of degree less than $4n - 1$ and $4n + 1$, respectively.

Let $T_n(x) = \text{Cos}\,(n \text{ arc Cos } x)$, $n = 0, 1 \ldots$ be the Chebyshev polynomials of the first kind defined on $[-1, 1]$. If $f(x)$ is continuous and bounded variation on $[-1, +1]$, then f has a uniformly convergent Chebyshev-Fourier expansion over $[-1, 1]$:

$$f(x) = \sum_{n=0}^{\infty}{}' a_n T_n(x), \tag{3}$$

(The prime on the summation indicates that the first term is to be halved). With $x = \text{Cos}\,\theta$, since $T_n(x) = \text{Cos}\, n\theta$, (3) becomes

$$f(\cos\theta) = \sum_{n=0}^{\infty}{}' a_n \cos n\theta, \tag{4}$$

the Fourier cosine expansion of the even periodic function $f(\text{Cos}\,\theta)$ over half the period $[0, \pi]$. The coefficients in the Fourier expansion (3) or (4) are given exactly as

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos n\theta\, d\theta, \quad n = 0, 1, \ldots \tag{5}$$

Let

$$L(f) = \int_0^{\pi} f(\cos\theta)\, d\theta = \frac{\pi}{2} a_0. \tag{6}$$

In practice, the integrals are approximated by sums over a half period. The trapezoidal rule is approximated using N or $N + 1$ points

$$\theta_j = (j + \omega)\frac{\pi}{N}, \quad j = 0(1)N-1, \quad \text{for } \omega = \frac{1}{2}$$
$$= 0(1)N, \qquad \text{for } \omega = 0.$$

Our main results in the paper are based on a theorem in section 2 connecting interpolating coefficients to the Fourier coefficients. In section 3, we then show that certain closed form Kronrod type quadratures with their errors turn out to coincide with the extended Gauss Chebyshev quadratures. In Sec. 4 we obtain explicitly a family of extended rules round the unit circle $C_1 : |Z| = 1$. Certain extended rules over $[-1, 1]$ with the weight function $(1 - u^2)^{-1/2}$ have been derived from the extended rules over C_1, for if $f(w)$ is analytic on $[-1, 1]$, then $f\left(\frac{z + z^{-1}}{2}\right) \in A(R(r^{-1}, r)), r > 1$. Using Laurants expansion, the estimates of these rules more simply turn out to be the same as would otherwise result over Hilbert spaces through Davis method.

2. Determination of Chebyshev coefficients. For $w = 1/2$, we required to introduce

$$f(x_j) = \sum_{i=0}^{N-1}{}' \alpha_i T_i(x_j),$$

given the functional values at N points $x_j = \cos\left(j + \frac{1}{2}\right)\frac{\pi}{N}$, $j = 0(1)N-1$. For $w = 0$, we need instead $f(x_j') = \sum_{i=0}^{N}{}'' \alpha_i' T_i(x_j')$, given that the function values at the $N + 1$ points $x_j' = \cos\frac{j\pi}{N}$, $j = 0(1)N$. (Here single prime indicates that the first term is to be halved and double prime means that first and the last term is to be multiplied by 1/2).

Since $\theta_i = (2i - 1)\frac{\pi}{2N}$, $i = 1(1)N$ be n points equi-spaced inside $[0, \pi]$, so that $x_k = \text{Cos}\,\theta_k$ are the zeroes of $T_k(x)$ over $[-1, 1]$. Consider the mid-point approximation

$$\frac{\pi}{2}\alpha_0 = \frac{\pi}{N}\sum_{k=1}^{N} f(x_k), \tag{7}$$

for $L(f)$. If we define the numbers

$$\alpha_i = \frac{2}{N}\sum_{j=1}^{N} f(x_j)\, T_i(x_j), \quad i = 1(1)N, \quad \omega = \frac{1}{2}, \tag{8}$$

and

$$\alpha_i' = \frac{2}{N}\sum_{j=0}^{N}{}'' f(x_j')\, T_i(x_j'), \quad i = 0(1)N, \quad \omega = 0. \tag{9}$$

It is of interest to compare the interpolation coefficients α_i, α_i' with the Fourier coefficients a_i so that

$$\alpha_i = a_i + \sum_{m=1}^{\infty}(-1)^m(a_{2mN-i} + a_{2mN+i}), \quad i = 0(1)N-1 \tag{10}$$

and

$$\alpha_i' = a_i + \sum_{m=1}^{\infty}(a_{2mN-i} + a_{2mN+i}), \qquad i = 0(1)N. \tag{11}$$

(See Fox and Parker [1972]). From the above two equations, we have:

THEOREM 1. *If α_i and α_i' are the Chebyshev Coefficients and a_i are the usual Fourier Coefficients as defined above, then*

$$\frac{1}{2}(\alpha_i + \alpha_i') = a_i + a_{4N-i} + a_{4N+i} + \ldots \tag{12}$$

We note that above is a better approximation of a_i for all i, $0 \leqslant i \leqslant N$ as compared to (10) or (11).

3. Derivation of quadrature formulae

1. Extended Gauss-Chebyshev Quadrature Formulae (closed type). For $i = 0$, Eq. (12) gives

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x)\mathrm{d}x = \frac{\pi}{N} \left[\sum_{j=0}^{N-1} f\left(\cos (2j - 1) \frac{\pi}{2N} \right) + \right.$$

$$\left. + \sum_{j=0}^{N}{}'' f\left(\cos \frac{j\pi}{N} \right) \right] + E^{kE}_{P_{2N+1}}(f). \tag{13}$$

We note that above is Kronrod-type quadratures of the form (2). Upon simplification, we have

THEOREM 2.

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x)\mathrm{d}x = \frac{\pi}{2N} \sum_{j=0}^{2N}{}'' f\left(\cos \frac{j\pi}{2N} \right) + E^{kE}_{P_{2N+1}}(f). \tag{13'}$$

Meanwhile, we find that above is simply the Kronrod extension (KE) of the n-point GJIR in the closed form corresponding to $\alpha = \beta = -1/2$ as shown by Monegato [1976]. Upon comparison with (12), the corresponding error is given by

$$E^{kE}_{P_{2N+1}}(f) = \pi \sum_{m=1}^{\infty} a_{4mN}. \tag{14}$$

We observe that (14) implies that $E^{kE}_{P_{2N+1}}(f) = 0$, whenever f is a polynomial of degree less than $4N$. If the Chebyshev coefficients for 'f' decrease sufficiently rapidly, then for large N,

$$E^{kE}_{P_{2N+1}}(f) \simeq \pi a_{4N}. \tag{14'}$$

above indicates that the above rule is exact for polynomials of degree $4N - 1$.

Case (ii) Extended Gauss-Chebyshev quadrature formula (2ND kind). The extended Gauss-Chebyshev quadrature formula of the closed type (13) applied to the function $(1 - x^2)f(x)$ with $2N + 3$ point based on Kronrod-rule gives

THEOREM 3.

$$\int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} f(x)\mathrm{d}x = \frac{\pi}{2(N+1)} \left[\sum_{j=1}^{N+1} \sin^2 (2j - 1) \frac{\pi}{2N+2} \times \right.$$

$$\left. \times f\left(\cos \frac{2j-1}{2N+2} \pi \right) + \sum_{j=1}^{N} \sin^2 \left(\frac{j\pi}{N+1} \right) f\left(\cos \frac{j\pi}{N+1} \right) \right] + E^{kE}_{U_{2N+1}}(f) =$$

$$= \frac{\pi}{2N+2} \sum_{j=1}^{2N+1} (1 - x_j^2) f(x_j) + E^{kE}_{U_{2N+1}}(f), \tag{15}$$

where $x_j = \mathrm{Cos} \dfrac{j\pi}{2N+2}$. The corresponding error is given by

$$E^{kE}_{U_{2N+1}}(f) = E^{kE}_{T_{2N+3}}((1 - x^2)f(x)).$$

If a_n^* denotes the Chebyshev Fourier coefficients for $(1 - x^2)f(x)$, then

$$a_n^* = \frac{1}{4}(2a_n - a_{n+2} - a_{|n-2|}), \quad n = 0, 1, \ldots$$

Since
$$E^{kE}_{U_{2N+1}}(f) = \pi \sum_{m=1}^{\infty} a^*_{4(N+1)m}$$

$$= \frac{\pi}{4} \sum_{m=1}^{\infty} (a_{4mN+4m-2} - 2a_{4mN+4m} + a_{4mN+4m+2}). \tag{16}$$

The above indicates that the above rule is exact for the polynomials of degree $4N + 1$.

(iii) Extended Gauss-Jacobi quadrature formula (semi-open type). The extended quadrature formula of the type (13) applied to the function $(1 - x)f(x)$ with $2N + 1$ points gives

THEOREM 4.

$$\int_{-1}^{1} \left(\frac{1-x}{1+x} \right)^{1/2} f(x)\mathrm{d}x = \frac{\pi}{N} \left[\sum_{j=0}^{N-1} \sin^2 (2j + 1) \frac{\pi}{4N} \cdot f\left(\cos \frac{2j+1}{2N} \pi \right) + \right.$$

$$\left. + \sum_{j=0}^{N}{}'' \sin^2 \frac{j\pi}{2N} f\left(\cos \frac{j\pi}{N} \right) \right] + E^{kE}_{J_{2N+1}} (1 - x) f(x).$$

$$= \frac{\pi}{2N} \sum_{j=0}^{2N}{}'' (1 - x_j) f(x_j) + E^{kE}_{J_{2N+1}}(f). \tag{17}$$

where $x_j = \mathrm{Cos} \dfrac{j\pi}{2N+2}$ and the last term in the summation above is to be multiplied by 1/2. Above is Semi-open formula because of weight zero at $x = 1$. It is easy to obtain

$$E^{kE}_{J_{2N+1}}(f) = \frac{\pi}{2} \sum_{m=1}^{\infty} (a_{4mN-1} - 2a_{4mN} + a_{4mN+1}). \tag{18}$$

With the help of (18), (17) is exact for all polynomials of degree $\leqslant 4N - 2$.

Now, let ε_r denote the closed elliptic disk in the complex plane bounded by the ellipse with foci at $(1, 0)$ and $(-1, 0)$ and with half axes a and b, where $a + b = r > 1$. Let f be real valued on $[-1, 1]$ with an extension which is analytic on ε_r. Then it is well known (Cf. Meinardus [11, p. 91]) :

$$|a_k| \leqslant \frac{2M_r}{r^k}, \quad M_r = \sup\{|f(z)| z \in \varepsilon_r\}. \tag{19}$$

Thus, with the help of (19) from (14), (16) and (18), we can find the estimates of errors of the corresponding formulae.

4. Extended integration rule round the unit circle. If $f(u)$ $(u = R(w)$ is analytic on $[-1, 1]$, then there exists $r > 1$ such that $f(w) \in A(\varepsilon_\rho)$ and, subsequently, $f\left(\dfrac{Z+Z^{-1}}{2}\right) \in A(R(r^{-1}, r))$. Since the unit circle $C_1(R(r^{-1}, r))$, $r > 1$ and under transformation $w = 1/2\ (Z + Z^{-1})$, C_1 is mapped onto the interval $-1 \leqslant w \leqslant 1$ counted twice, an extended rule over C_1 is therefore obtained from an equivalent rule over $[-1, 1]$.

Now, let $f(w) \in A(\varepsilon_r)$, $r > 1$ and $F(z) = f\left(\dfrac{z+z^{-1}}{2}\right)$, then

$F(z) \in A(r^{-1}, r)$, $r > 1$. Since

$$\frac{1}{2}\int_{C_1} F(z)ds = \int_{-1}^{1} (1-u^2)^{-1/2} f(u)du = I(f). \tag{20}$$

From (13′) and (20), we have

$$2E^{kE}_{P_{2N+1}}(f) = E^{kE}_{P_{2N+1}}(F) = \int_{C_1} F(z)ds - \frac{\pi}{2N}\sum_{j=0}^{4N-1} F(e^{\frac{\pi j i}{N}}). \tag{21}$$

We observe that above is a particular case of Theorem 5.

THEOREM 5.

$$E^{kE}_n(F) = \int_{C_1} F(z)\,ds - \sum_{j=0}^{2N-1} \frac{\pi}{n} F(e^{i(\alpha+\pi j/N)}), \tag{22}$$

for $\alpha = 0$ *and* $n = 2N$.

The above represents an extended family of integration rules over C_1. We now find the estimates for (22). Applying Laurants expansion to $F(z)$, we have

$$F(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^{-k}, \tag{23}$$

where

$$a_k = \frac{1}{2\pi i}\int_{C_r} \frac{F(t)}{t^{k+1}}\,dt \ \&\ b_k = \frac{1}{2\pi i}\int_{C_{r^{-1}}} t^{k-1}F(t)dt.$$

If E_n denotes the error of some numerical approximation, we have from (23)

$$E_n(F(z)) = \sum_{k=0}^{\infty} a_k E_n(z^k) + \sum_{k=1}^{\infty} b_k E(z^{-k}), \tag{24}$$

or $$|E_n(F(z))| \leqslant \sum_{k=0}^{\infty} |a_k|\,|E_n(z^k)| + \sum_{k=1}^{\infty} |b_k|\,|E_n(z^{-k})|.$$

Noting $|a_k| \leqslant r^{-k}M_r$ and $|b_k| \leqslant r^{-k}M_{r^{-1}}$,

where $M_r = \max |F(z)|$ on $|z| = r$.

Since $E_n(z^k) = \begin{cases} -2\pi^{i\alpha mn}, & k = \pm 2n, \pm 4n, \ldots \\ 0, & k \neq \pm 2n, \pm 4n, \ldots \end{cases}$

Now substituting the above in (24), we have

$$|E_n(F)| \leqslant \frac{2\pi}{r^{2n}-1}(M_r + M_{r^{-1}}). \tag{25}$$

We remark that a proper choice of α in $\left[0, \dfrac{\pi}{2N}\right]$ may help to reduce the number of function evaluations. In particular, we note the following extended quadratures.

For $\alpha = 0$, $n = 2N + 1$,

$$E1^{kE}_{2N+1}(f) = I(f) - \frac{\pi}{4N+1}\left(f(1) + 2\sum_{j=1}^{2N} f\left(\cos\frac{2j\pi}{4N+1}\right)\right). \tag{26}$$

For $\alpha = \pi/n$, $n = 2N + 1$,

$$E2^{kE}_{2N+1}(f) = I(f) - \frac{\pi}{4N+1}\left(f(-1) + 2\sum f\left(\cos\frac{(2j-1)k}{4N+1}\right)\right). \tag{27}$$

For $\alpha = \pi/n$, $n = 2N$,

$$E^{kE}_{T_{2N+1}}(f) = I(f) - \frac{\pi}{2N}\sum_{j=1}^{2N} f\left(\cos(2j-1)\frac{\pi}{4N}\right), \tag{28}$$

The error estimates for (26–28) at once follow from (25). Further we observe that using (25) the same estimates do result for (14), (16) and (18) as derived in section 3. All these estimates turn out more simply to be the same as otherwise result using different Hilbert spaces via Davis method.

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