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ON SOME INEQUALITIES INVOLVING ISOTONIC FUNCTIONALS

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Introduction. Let E be a nonempty set, $\mathcal{F}(E, R)$ the algebra of all functions $f: E \to R$ and $\mathcal{A}$ a subalgebra of $\mathcal{F}(E, R)$. A linear functional $A: \mathcal{A} \to R$ is called isotonic if the following condition is satisfied : $f \in \mathcal{A}$, $f \geqslant 0$ implies $A(f) \geqslant 0$. Let us denote by $M_+^1(\mathcal{A}, R)$ the set of all linear isotonic functionals normalized by the condition $A(1) = 1$, where $1 \in \mathcal{A}$ is the unity of $\mathcal{A}$.

Some inequalities for the functionals in $M_+^1(\mathcal{A}, R)$ were investigated by many authors in different situations for the set E. B. Jessen [5] generalizes the well-known Jensen's inequality for convex functions, P. R. Beesack and J. E. Pečarić [4], [11] extend some classical inequalities, J. E. Pečarić and D. Andrica [9], [10] study some Jessen's type inequalities in abstract spaces and Grüss' type inequalities, D. Andrica and C. Badea [3] give some generalizations for the inequality of Grüss.

In the first section of this paper we give two-new inequalities for functionals in $M_+^1(\mathcal{A}, R)$ and functional determinants. One of this inequalities is a generalization of the well-known Gram's inequality (see [7] pp. 45). In the second section we improve the results given in [1], [6] for convex functions. As an application we obtain a Korovkin type theorem for Markov operators.

Inequalities for functional determinants. In the following we consider the functions $f_{ij} \in \mathcal{A}$, $i, j = 1, 2, \ldots, n$, $f_{ij} = f_{ji}$. We assume that for every $t \in E$

(1) $$D_k(t) = \det (f_{ij}(t))_{1 \leqslant i,j \leqslant k} > 0, \quad k = 1, 2, \ldots, n$$

For $A \in M_+^1(\mathcal{A}, R)$ denote $A[D_k] = \det (A(f_{ij}))_{1 \leqslant i,j \leqslant k}$.

THEOREM 1. *If $A \in M_+^1(\mathcal{A}, R)$ and $A[D_k] \neq 0$, then*

(2) $$A[D_{k+1}]/A[D_k] \geqslant A(D_{k+1}/D_k)$$

Proof. Let us consider the function $F: E \times R^k \to R$, $F(t, x) = \sum_{i,j=1}^{k+1} f_{ij}(t)\, x_i\, x_j$, where $x_{k+1} = 1$. We observe that

(3) $$F(t, x) \geqslant \min_{x \in R^k} F(t, x) = D_{k+1}(t)/D_k(t)$$

Applying A to the inequality (3) we get:

$$\sum_{i,j=1}^{k+1} A(f_{ij})\, x_i x_j \geq A(D_{k+1}/D_k), \quad x=(x_1,\dots,x_k)\in R^k$$

It follows

$$\min_{x\in R^k}\sum_{i,j=1}^{k+1} A(f_{ij})\, x_i x_j \geq A(D_{k+1}/D_k)$$

and we obtain the inequality (2).

COROLLARY 1. *If* $A\in M_+^1(\mathscr{A},R)$, *then*

(4) $$(A[D_n])^{1/n} \geq A(D_n^{1/n})$$

Proof. For the beginning we suppose that $A(D_k)\neq 0$, $k=1,2,\dots,n$. Using (2) and the extended Hölder inequality (see [4]) we get

$$A[D_n] = A[D_1]\frac{A[D_2]}{A[D_1]}\cdot\frac{A[D_3]}{A[D_2]}\cdots\frac{A[D_n]}{A[D_{n-1}]} \geq A(D_1)A(D_2/D_1)\dots$$

$$\dots A(D_n/D_{n-1}) = A((\sqrt[n]{D_1})^n)\cdot A((\sqrt[n]{D_2/D_1})^n)\dots$$

$$\dots A((\sqrt[n]{D_n/D_{n-1}})^n) \geq A^n\left(\sqrt[n]{D_1\frac{D_2}{D_1}\cdots\frac{D_n}{D_{n-1}}}\right) = A^n(\sqrt[n]{D_n}).$$

In the case when the conditions $A[D_k]\neq 0$, $k=1,2,\dots,n$ are not satisfied we consider the functions $f^*_{ij}(t)=f_{ij}(t)+\varepsilon_{ij}$, where $\varepsilon_{ij}\in R$, and $A[D_k^*]\neq 0$, $k=1,2,\dots,n$.

We have $A(f^*_{ij}) = A(f_{ij})+\varepsilon_{ij}$ and it follows that $A[D_n^*] = A[D_n]+\varepsilon$, where $\varepsilon\to 0$ when $\varepsilon_{ij}\to 0$. According to the above result we get $(A[D_n^*])^{1/n}\geq A(D_n^{*1/n})$ and for $\varepsilon_{ij}\to 0$ one obtains the inequality (4).

COROLLARY 2. *If* $A\in M_+^1(\mathscr{A},R)$ *and* $A[D_n]\neq 0$, *then*

(5) $$1/A[D_n] \leq A(1/D_n)$$

Proof. Supposing that $A[D_k]\neq 0$, $k=1,2,\dots,n$ we observe that

$$\frac{1}{A[D_n]} = \frac{1}{A[D_1]}\cdot\frac{A[D_1]}{A[D_2]}\cdots\frac{A[D_{n-1}]}{A[D_n]} \leq \frac{1}{A(D_1)}\cdot\frac{1}{A\left(\frac{D_2}{D_1}\right)}\cdots$$

$$\cdots\frac{1}{A\left(\frac{D_n}{D_{n-1}}\right)} \leq A\left(\frac{1}{D_n}\right)$$

because from the extended Hölder inequality [4] we have

$$A_1(D_1)A\left(\frac{D_2}{D_1}\right)\cdots A\left(\frac{D_n}{D_{n-1}}\right)\cdot A\left(\frac{1}{D_n}\right)\geq A^{n+1}\left(\sqrt[n+1]{D_1\frac{D_2}{D_1}\cdots\frac{D_n}{D_{n-1}}\cdot\frac{1}{D_n}}\right) = A(1)=1.$$

If the conditions $A[D_k]\neq 0$, $k=1,2,\dots,n$ are not satisfied we consider the functions $f^*_{ij}(t)=f_{ij}(t)+\varepsilon_{ij}$, as in the proof of Corollary 1.

AN APPLICATION. In the case $n=2$ and $A\in M_+^1(\mathscr{A},R)$, $A(f)=\frac{1}{2}(f(x_1)+f(x_2))$ from Corollary 2, with $f_{11}=f$, $f_{12}=f_{21}=g$, $f_{22}=h$, we obtain

$$\frac{1}{\begin{vmatrix}\frac{1}{2}(f(x_1)+f(x_2)) & \frac{1}{2}(g(x_1)+g(x_2))\\ \frac{1}{2}g(x_1)+g(x_2)) & \frac{1}{2}(h(x_1)+h(x_2))\end{vmatrix}} \leq \frac{1}{2}\,\frac{1}{\begin{vmatrix} f(x_1) & g(x_1)\\ g(x_1) & h(x_1)\end{vmatrix}}+\frac{1}{\begin{vmatrix} f(x_2) & g(x_2)\\ g(x_2) & h(x_2)\end{vmatrix}}$$

Let $a_i=f(x_i)$, $b_i=g(x_i)$, $c_i=h(x_i)$, $i=1,2$ and it results the inequality

(6) $$\frac{8}{(a_1+a_2)(c_1+c_2)-(b_1+b_2)^2}\leq\frac{1}{a_1c_1-b_1^2}+\frac{1}{a_2c_2-b_2^2}$$

under the conditions $a_i>0$, $a_ic_i-b_i^2>0$, $i=1,2$.

The inequality (6) is the problem 3.31 of [8].

THEOREM 2. *If* $A\in M_+^1(\mathscr{A},R)$ *and* (1) *is true then*

(7) $$A[D_p]\geq 0,\quad p=1,2,\dots,n$$

Proof. From (1) we obtain $\sum_{i,j=1}^{p} f_{ij}(t)\,x_ix_j\geq 0$ for every $x\in R^p$ and for every $t\in E$. Applying A we get $\sum_{i,j=1}^{p} A(f_{ij})\,x_ix_j\geq 0$ for every $x=(x_1,\dots,x_p)\in R^p$.

According to the theorem of Sylvester we obtain (7).

An inequality for the functions in $C^2(E)$. We consider E a compact convex subset with nonempty interior of the Euclidean space R^n. Let $p_k: E\to R$ be the k-projection, defined by $p_k(x_1,x_2,\dots,x_n)=x_k$, $k=1,2,\dots,n$. The following result is given in [9].

THEOREM 3. *If* $f\in C(E)$ *is convex on* E *then for every functional* $A\in M_+^1(C(R),R)$ *the following inequality holds:*

(8) $$A(f)\geq f(A(p_1),\dots,A(p_n))$$

Our next result improves the inequality (8) in the case when f is a function in $C^2(E)$, not necessary convex on E. In this case we consider the Hessian matrix of f defined by

(9) $$H(f)(x)=\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right)_{1\leq i,j\leq n},\quad x\in E$$

Denote by $\|H(f)(x)\|$ the Frobenius norm of $H(f)(x)$ and by $K_f = \max_{x \in E} \| H(f)(x) \|$.

THEOREM 4. *If* $f \in C^2(E)$ *then*

$$|A(f) - f(A(p_1), \ldots, A(p_n))| \leqslant K_f \sum_{k=1}^{n} [A(p_k^2) - A^2(p_k)] \tag{10}$$

Proof. According to the Taylor formula we have

$$f(x) = f(y) + \sum_{k=1}^{n} (x_k - y_k) \frac{\partial f}{\partial x_k}(y) + \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi)(x_i - y_i)(x_j - y_j)$$

so that

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle = \langle (Hf)(\xi)(x - y), x - y \rangle$$

where, $\langle , \rangle$ denotes the inner product in R^n. From the Schwarz inequality we get

$$|f(x) - f(y) - \langle \nabla f(y), x - y \rangle| = |\langle H(f)(\xi)(x - y), x - y \rangle| \leqslant$$
$$\| (H(f)(\xi)(x - y) \| \, \|x - y\| \leqslant \|H(f)(\xi)\| \, \|x - y\|^2 \leqslant K_f \|x - y\|^2.$$

It results

$$-K_f \|x - y\|^2 + \langle \nabla f(y), x - y \rangle \leqslant f(x) - f(y) \leqslant \langle \nabla f(y), x - y \rangle + + K_f \|x - y\|^2$$

Applying A with respect to x we obtain

$$- K_f \sum_{k=1}^{n} (A(p_k^2) - 2y_k A(p_k) + y_k^2) + \sum_{k=1}^{n} (A(p_k) - y_k) \frac{\partial f}{\partial y_k}(y) \leqslant \tag{11}$$
$$\leqslant A(f) - f(y) \leqslant \sum_{k=1}^{n} (A(p_k) - y_k) \frac{\partial f}{\partial x_k}(y) +$$
$$+ K_f \sum_{k=1}^{n} (A(p_k^2) - 2y_k A(p_k) + y_k^2)$$

Taking $y_k = A(p_k)$, $k = 1, 2, \ldots, n$ from (11) we get (10).

In the case $n = 1$ from (10) we obtain a result contained in [1].

The linear operator $L : C(E) \to C(E)$ is called a Markov operator if $L(1) = 1$ and $L(f) = f$ for every linear function f on E.

COROLLARY 3. *Let* (L_α) *be a sequence of Markov operators on* $C(E)$ *If* $L_\alpha(e) \xrightarrow{u} e$, *where* $e(x) = \| x \|^2$, *then* $L_\alpha(f) \to f$ *for every function* $f \in C(E)$.

Proof. From (10) with $A(f) = L_\alpha(f)(x)$ we get $|L_\alpha(f)(x) - f(x)| \leqslant K_f |L_\alpha(e)(x) - e(x)|$. So it results $\|L_\alpha(f) - f\| \leqslant K_f \|L_\alpha(e) - e\|$ for every function $f \in C^2(E)$, where $\| \cdot \|$ is the supremum norm in $C(E)$. From the above nequality it follows that $L_\alpha(f) \to f$ for every $f \in C^2(E)$. But $C^2(E)$ is a dense subspace of $C(E)$ and the corollary is proved via the Banach-Steinhauss theorem, because it is eassy to see that $\|L_\alpha\| = 1$.

In the case $n = 2$ we obtain a result due to A. Lupaş [6] and V. I. Volkov [12].

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