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DISCRETE CONVEXITY CONES

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1. Let us consider the linear recurrence of order p :

$$L_p(x_n) = \sum_{j=0}^{p} d_j\, x_{n+j} = 0,\ n \geqslant 0 \tag{1}$$

where $d_p = 1$ and $d_0 \neq 0$. As it is known (see [2]), the representation of the sequences which satisfy this relation is related to the solutions of the algebraic equation :

$$L_p(t^n)/t^n = \sum_{j=0}^{p} d_j\, t^j = \prod_{j=1}^{p} (t - t_j). \tag{2}$$

For example, we shall use the sequence $(u_n)_{n \geqslant 0}$ defined by :

$$L_p(u_n) = 0,\ \forall n \geqslant 0\ ;\ u_0 = \ldots = u_{p-2} = 0,\ u_{p-1} = 1. \tag{3}$$

If the roots of (2) are s_i multiple of order q_i, for $i = 1, \ldots, r$ (with $q_1 + \ldots + q_r = p$), then :

$$u_n = \sum_{i=1}^{r} P_i(n) \cdot s_i^n$$

where P_i is a polynomial of degree q_i and

$$\sum_{i=1}^{r} P_i(j) \cdot s_i^j = u_j \quad \text{for } j = 0, \ldots, p-1.$$

So, if $r = 1$, that is $t_1 = \ldots = t_p = s$, then :

$$u_n = s^n \cdot \binom{n}{p-1}$$

and if $r = p$, that is $t_i \neq t_j$ for $i \neq j$, then :

$$u_n = \sum_{j=1}^{p} \left[t_j^n \Big/ \prod_{\substack{i=1 \\ i \neq j}}^{p} (t_j - t_i) \right],$$

References to other methods of representation of recurrent sequences may be found in [1].

Our basic method of study is furnished by the following result which may be proved by simple computation (see [16]):

LEMMA 1. *If the sequence* $(x_n)_{n\geq 0}$ *is represented by:*

$$x_n = \sum_{i=0}^{n} u_{n+p-i-1}\, y_i \tag{4}$$

where $(u_n)_{n\geq 0}$ *is given by* (3), *then:*

$$L_p(x_n) = y_{n+p}.$$

If $(x_n)_{n\geq 0}$ is given, then $(y_n)_{n\geq 0}$ may be found, step by step, from (4), so that we get:

LEMMA 2. *Let* $p \subset \mathbb{R}$. *In order that* $L_p(x_n) \in P$ *for every* $n \geq 0$ *it is necessary and sufficient that* $(x_n)_{n\geq 0}$ *be represented by* (4) *with* $y_i \in P$ *for* $i \geq p$.

COROLLARY 1. *The sequence* $(x_n)_{n\geq 0}$ *verifies the relations:*

$$L_p(x_n) = z_n, \quad n \geq 0$$

if and only if it is represented by (4) *with* $y_i = z_{i-p}$ *for* $i \geq p$.

COROLLARY 2. *The sequence* $(x_n)_{n\geq 0}$ *verifies the relation* (1) *if and only if it is represented by:*

$$x_n = \sum_{i=0}^{p} u_{n+p-i-1}\; y_i\,. \tag{5}$$

On the vector space S of all sequences, let us consider the shift operator E defined for any $x = (x_n)_{n\geq 0}$ by:

$$Ex = x' = (x'_n)_{n\geq 0}, \quad x'_0 = 0, \quad x'_n = x_{n-1}, \quad n \geq 1.$$

If we define the sequence:

$$u^p = (u_{p-1+n})_{n\geq 0} \tag{5}$$

the relation (5) may be rewritten as:

$$x = \sum_{i=0}^{p-1} y_i \cdot E^i u^p$$

where $E^0 x = x$ and E^i is obtained by the composition of i exemplars of E. Thus we have:

COROLLARY 3. *The sequences:*

$$u^p, Eu^p, \ldots, E^{p-1}u^p$$

form a basis for the subspace of sequences which verify (1).

2. In what follows, we shall deal with the cone of convex sequences in respect to the operator L_p, that is:

$$K_m(L_p) = \{(x_n)_{n=0}^{m} : L_p(x_n) \geq 0,\ 0 \leq n \leq m-p\}$$

or

$$K(L_p) = \{(x_n)_{n\geq 0} : L_p(x_n) \geq 0,\ n \geq 0\}.$$

The case $t_1 = \ldots = t_p = 1$ corresponds to the usual convexity of order p as $L_p = \Delta^p$ (see [12]). We have given the representation of these (ordinary) convex sequences in [15], for the case $p = 2$ (and L_2 arbitrary) in [9] and for the general case in [16]. This follows from Lemma 1.

THEOREM 1. *a) The sequence* $(x_n)_{n=0}^{m}$ *belongs to* $K_m(L_p)$ *if and only if it may be represented by* (4), *with* $y_i \geq 0$ *for* $p \leq 1 \leq m-p$.

b) The sequence $(x_n)_{n\geq 0}$ *belongs to* $K(L_p)$ *if and only if it may be represented by* (4) *with* $y_i \geq 0$ *for* $i \geq p$.

The result from part b) may be reformulated if we consider (as it was done in [5] and then in [10], [11] and [17]) the metric d on S, defined by:

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$. Let us also put:

$$L_p(x) = (L_p(x_n))_{n\geq 0}.$$

We have at once:

LEMMA 3. *If* u^p *is given by* (6) *then:*

$$L_p(E^k u^p) = 0 \quad \text{for} \quad 0 \leq k \leq p-1$$

and

$$L_p(E^k u^p) = (\delta_{n,k-p})_{n\geq 0} \text{ for } k \geq p$$

where $\delta_{n,k}$ *is Kronecker's symbol.*

THEOREM 2. *The sequence* x *belongs to* $K(L_p)$ *if and only if:*

$$x = \lim_{n\to\infty} x^n = \sum_{n=0}^{\infty} x^n \tag{7}$$

where

$$x^n = \sum_{k=0}^{n} y_k \cdot E^k u^p, \text{ with } y_k \geq 0 \text{ for } k \geq p$$

and the limit is taken in respect to the metric d.

Proof. As $E^n u^p$ has the first n components zero, any sequence x is the limit of such a linear combination (in fact, x and x^n have the same first $n+1$ components). But

$$L_p(x^n) = (y_p, \ldots, y_n, 0, 0 \ldots) \to L_p(x)$$

so that x is in $K(L_p)$ if and only if $y_n \geq 0$ for $n \geq p$.

3. In [16] we have also characterized the elements of the dual cone of $K_m(L_p)$ that is:

$$K_m^*(L_p) = \left\{ (a_n)_{n=0}^m : \sum_{k=0}^m a_k x_k \geq 0,\ \forall (x_k)_{k=0}^m \in K_m(L_p) \right\}.$$

As it is stated in [3], such results were obtained for the first time for convex functions by T. Popoviciu (see [14] for more references).

They were transposed for convex sequences by J. E. Pečarić in [13]. A constructive characterization is given in [20]. The representation for $p=2$ is given in [8]. The general case follows easy from Theorem 1.

THEOREM 3. *The sequence $(a_n)_{n=0}^m$ belongs to $K_m^*(L_p)$ if and only if it satisfies the relations:*

$$\sum_{n=k}^m a_n u_{n+p-k-1} = 0 \text{ for } 0 \leq k \leq p-1$$

and

$$\sum_{n=k}^m a_n u_{n+p-k-1} \geq 0 \text{ for } p \leq k \leq m.$$

Using Theorem 2 we can transpose the result for the case of m infinite. But, as in [17] we want to deal with a more general case. We remind some definitions. The functional $A : S \to \mathbb{R}$ is said to be:

a) superadditive, if:

$$A(x+y) \geq A(x) + A(y), \quad \forall x, y \in S;$$

b) positively superhomogeneous, if:

$$A(ax) \geq a \cdot A(x), \quad \forall x \in C, \forall a \geq 0;$$

c) upper semicontinuous, if:

(8) $$\limsup_{n\to\infty} A(x^n) \leq A(\lim_{n\to\infty} x^n).$$

THEOREM 4. *Let $A : S \to \mathbb{R}$ be a supperadditive, positively superhomogeneous, upper semicontinuous functional. In order that $A(x) \geq 0$ for every $x \in K(L_p)$ it is necessary and sufficient that:*

(9) $$A(E^k u^p) \geq 0 \text{ for } k \geq 0$$

and

(10) $$A(-E^k u^p) \geq 0 \text{ for } 0 \leq k < p.$$

Proof. From the theorem 2, we have $E^k u^p \in K(L_p)$ for $k \geq 0$ and also $-E^k u_p \in K(L_p)$ for $0 \leq k < p$, so that the conditions (9) and (10) are necessary. They are also sufficient. For an $x \in K(L_p)$ we have (7) and so, for $n > p$:

$$A(x^n) = A(y_0 u^p + y_1 \cdot Eu^p + \ldots + y_n \cdot E^n u^p) \geq A(y_0 u^p) +$$
$$+ A(y_1 \cdot Eu^p) + \ldots + A(y_n \cdot E^n u^p) \geq |y_0| \cdot A((\operatorname{sgn} y_0)\, u^p) +$$
$$+ \ldots + |y_{p-1}| \cdot A((\operatorname{sgn} y_{p-1}) \cdot E^{p-1} u^p) + y_p \cdot A(E^p u^p) +$$
$$+ \ldots + y_n \cdot A(E^n u^p) \geq 0$$

thus, from (8), $A(x) \geq 0$.

COROLLARY 4. *Let $A : S \to \mathbb{R}$ be a linear and continuous functional. In order that $A(x) \geq 0$ for every $x \in K(L_p)$ it is necessary and sufficient that:*

$$A(E^k u^p) = 0 \quad for \quad 0 \leq k < p$$

and

$$A(E^k u^p) \geq 0 \quad for \quad k \geq p.$$

We remark that in this corollary $\mathbb{R}$ can be replaced by an arbitrary linear topological space with a "positive" cone.

If we don't work with divergent series, Corollary 4 takes the following form. Let us denote:

$$K^*(L_p) = \left\{ a = (a_n)_{n\geq 0};\ \exists\, n_0 : a_n = 0 \text{ if } n > n_0 \ \&\ ax = \sum_{n=0}^{\infty} a_n x_n \geq 0,\ \forall x = (x_n)_{n\geq 0} \in K(L_p) \right\}.$$

COROLLARY 5. *The finally null sequence a belongs to $K^*(L_p)$ if and only if:*

$$a \cdot E^k u^p = 0 \quad for \quad 0 \leq k < p$$

and

$$a \cdot E^k u^p \geq 0 \quad for \quad k \geq p.$$

We point out that these results generalize the corresponding theorems from [5] and [17].

4. We can further generalize these results as follows. Let $A : S \to S$ be a continuous linear operator on S and L_p, L'_q two linear recurrences of the form (1). The problem is when holds:

(11) $$A(K(L_p)) \subset K(L_q).$$

THEOREM 5. *If $A : S \to S$ is a linear continuous operator, then* (11) *holds if and only if:*

$$L'_q(A(E^k u^p)) = 0 \quad for \quad 0 \leqslant k < p$$

and

$$L'_q(A(E^k u^p)) \geqslant 0 \quad for \ k \geqslant p.$$

Proof. As $E^k u^p \in K(L_p)$ for $k \geqslant 0$ and $-E^k u^p \in K(L_p)$ for $0 \leqslant k < p$, the conditions are necessary. They are also sufficiently. Indeed, let $x \in K(L_p)$. By (7), $x = \lim_{n\to\infty} x^n$, where $x^n = \sum_{k=0}^{n} y_k E^k u^p$ and $y_k \geqslant 0$ for $k \geqslant p$. So:

$$L'_q(A(x)) = \lim_{n\to\infty} L'_q(A(x_r)) =$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} y_k \cdot L'_q(A(E^k u^p)) = \lim_{n\to\infty} \sum_{k=p}^{n} y_k \cdot L'_q(A(E^k u^p)) \geqslant 0.$$

We remark that A is usually given by a double infinite matrix $A = (a_{nk})_{n,k\geqslant 0}$ with the property that for any $n \geqslant 0$ there is a k_n such that $a_{nk} = 0$ for $k > k_n$. If $x = (x_k)_{k\geqslant 0}$ then

$$A(x) = \left(\sum_{k=0}^{\infty} a_{nk} \ x_k\right)_{n\geqslant 0}.$$

The case of triangular matrices, that is $k_n = n$, was studied, for $L_p = L'_q = \Delta$ in [4] and [7]. His special case of generalized arithmetic means is effectively solved: the case $p = 2$ in [21] and in an improved form in [18], while the general case was initiated in [6] and accomplished in [19]. We shall give this result in the next paragraph. Also, the case $L_2 = L'_2$ is studied in [8].

5. Let $q = (q_n)_{n\geqslant 0}$ be a sequence of positive numbers. It defines an operator $Q : S \to S$ by: if $x = (x_n)_{n\geqslant 0}$ then $Q(x) = X = (X_n)_{n\geqslant 0}$ is given by:

$$X_n = (q_0 x_0 + \ldots + q_n x_n)/(q_0 + \ldots + q_n).$$

We denote by $K_p = K(\Delta^p)$ the set of (ordinary) p-convex sequences. In [19] we have proved that $Q(K_p) \subset K_p$ if and only if:

$$q_n = q_0 \binom{v+n-1}{n}, \quad n \geqslant 1 \tag{12}$$

with $v = q_1/q_0$, where:

$$\binom{w}{o} = 1, \ \binom{w}{n} = \frac{w(w-1)\ldots(w-n+1)}{n!} \ \text{for} \ n \geqslant 1.$$

Let us denote by $M^v K_p$ the set of sequences x with the property that $Q(x) \in K_p$, where q is given by (12). In [19] it is proved that $x \in M^v K_p$ if and only if:

$$x_n = \sum_{k=0}^{n} \binom{n+p-k-2}{p-2}\left(\frac{n+p-k-1}{p-1} + \frac{n}{v}\right) z_k, \ z_k \geqslant 0 \ \text{for} \ k \geqslant p.$$

This may be transcript as follows:

LEMMA 4. *The sequence x belongs to $M^v K_p$ if and only if:*

$$x = \sum_{k=0}^{\infty} \left[\left(1 + \frac{p-1}{v}\right) E^k u^p + \frac{k-p+1}{v} \ E^k u^{p-1}\right] z_k, \ z_k \geqslant 0 \ for \ k \geqslant p.$$

As in the other cases this gives:

THEOREM 6. *The linear continuous functional $A : S \to \mathbb{R}$ verifies the condition $A(x) \geqslant 0$ for every $x \in M^v K_p$ if and only if:*

$$(v+p-1)A(E^k u^p) + (k-p+1)A(E^k u^{p-1}) = 0 \ for \ 0 \leqslant k < p$$

and

$$(v+p-1)A(E^k u^p) + (k-p+1) \cdot A(E^k u^{p-1}) \geqslant 0 \ for \ k \geqslant p.$$

In the special case $p = 2$, $v = 1$ this result is given in [11].

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