MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 17, N° 1, 1988, pp. 73–91

ON SOME PROCEDURES FOR SOLVING FRACTIONAL MAX-MIN PROBLEMS

ŞTEFAN ŢIGAN
(Cluj-Napoca)

1. Introduction. In this paper we study several variants of the parametrical procedure [15] for solving fractional max-min programming problems. This procedure represents a generalization for the fractional max-min problems of the Dinkelbach parametrical method [7] given for usual fractional programming. It includes also the parametrical procedures considered by Schaible [12] and Tigan [16] for discrete max-min problems. In the particular case of the bilinear fractional max-min programming, a connection between the parametrical procedure and the substitution method [17] is established.

The paper is divided into seven sections. The main definitions, notations and the general statement of the fractional max-min problem are given in the second section.

Section 3 deals with a family of auxiliary nonfractional max-min problems associated to the fractional max-min problem. Several optimality conditions using the optimal value function of this family are derived. These results generalize similar properties obtained in the case of usual fractional programming problems (see, e.g. Refs. [4], [7], [8], [11], [14]).

Section 4 is devoted to the parametrical procedure for solving fractional max-min problems. Two parametrical iterative algorithms are given and a result concerning their convergence is proved. Both algorithms involve the solving of a (finite or infinite) sequence of auxiliary nonfractional max-min problems.

In Section 5 approximate finite variants of the parametrical procedure are considered.

Applications of the general parametrical procedure in certain particular fractional max-min problems are given in the last two sections. In the particular cases considered it is possible to simplify the general procedure by replacing, at each iteration, the solving of the auxiliary max-min problem either by an usual maximization problem in the case of separable fractional max-min problems (Section 6), or by a linear programming problem in the case of bilinear fractional max-min problem (section 7).

Other applications of the parametrical procedure to certain discrete fractional max-min problems are given in Refs. [10] and [20].

Also, by performing the Charnes-Copper variable change [5] in the bilinear fractional max-min programming problem with linear constraints, it can be shown that the specialization of the parametrical algorithm to this particular max-min problem can be regarded as a method for solving a certain nonlinear max-min programming problem with a quadratic objective function and nonlinear constraints of a special type.

2. Preliminary Notations, Definitions and Properties. Let $X \subseteq R^n$ and $Y \subseteq R^m$ be two nonvoid sets and let T be a point-to-set mapping from X to Y such that for every $x \in X$ the image set $T(x)$ is nonvoid. Let f and g be two real functions defined on the set $X \times Y$. Moreover, let us assume that

$$g(x, y) \neq 0 \text{ for all } (x, y) \in X \times Y. \tag{2.1}$$

Then we can define the fractional function $h : X \times Y \to R$ by

$$h(x, y) = \frac{f(x, y)}{g(x, y)} \text{ for every } (x, y) \in X \times Y. \tag{2.2}$$

The fractional max-min programming problem under consideration is:

FMM. Find

$$V = \max_{x \in X} \min_{y \in T(x)} \frac{f(x, y)}{g(x, y)}. \tag{2.3}$$

Definition 2.1. *A pair $(x', y') \in X \times Y$ is called an optimal solution for the fractional* max-min *problem FMM iff the following conditions are satisfied:*

$$y' \in T(x'), \tag{2.4}$$

$$h(x', y') = V, \tag{2.5}$$

$$\min_{y \in T(x')} h(x', y) = h(x', y'). \tag{2.6}$$

***Definition* 2.2.** *Let ε be a given nonnegative real number. A pair $(x', y') \in X \times Y$ is called ε-optimal solution for the fractional* max-min *problem FMM iff besides condition* (2.4) *the following two conditions are satisfied:*

$$h(x', y') + \varepsilon \geq V, \tag{2.7}$$

$$h(x', y') - \varepsilon \leq \min_{y \in T(x')} h(x', y). \tag{2.8}$$

It should be noted that the notion of ε-optimal solution introduced by Definition 2.2 generalizes that of optimal solution. Thus the set of ε-optimal solutions, besides the optimal solutions (i.e. 0-optimal solutions) includes approximate optimal solutions.

Throughout the paper concerning problem *FMM* we will assume that the following conditions hold:

*H*1) $T(x)$ is a compact set for every $x \in X$.
*H*2) X is a compact set.

*H*3) $g(x, y) > 0$ for every $x \in X$ and $y \in T(x)$.
*H*4) f and g are continuous functions on the set $X \times Y$.
*H*5) The point-to-set mapping T is continuous on X in the sense of Berge [3].

The above assumptions are satisfied, for instance, by the max-min fractional problems considered in Refs. [6], [2], [10], [17] and [19].

Under the assumptions *H*1) — *H*5), we can define the function $H : X \to R$ by

$$H(x) = \min_{y \in T(x)} h(x, y) \quad \text{for all } x \in X. \tag{2.9}$$

Concerning the function H we can state the following result.

LEMMA 2.1. *If the assumptions H*1) — *H*5) *hold, then H is a continuous function.*

Proof. The conclusion of the Lemma is a direct consequence of the "Theorem of maximum" (see Berge [3]).

From Lemma 2.1 and the assumptions *H*1) — *H*5) we can easily get the following result.

LEMMA 2.2. *The problem FMM has at least an optimal solution.*

Proof. The asserted result directly follows from the continuity of the functions h and H and the compactness of the sets X and $T(x)$ $(x \in X)$.

3. The Auxiliary Parametrical Problem and Related Properties. In this section we deal mainly with an auxiliary max-min problem associated with the fractional max-min problem *FMM*. This auxiliary problem depends on a real parameter t and can be stated as follows:

PA(t). Find

$$F(t) = \max_{x \in X} \min_{y \in T(x)} (f(x, y) - tg(x, y)). \tag{3.1}$$

Let us define the function $q : R \times X \times Y \to R$, by

$$q(t, x, y) = f(x, y) - tg(x, y). \tag{3.2}$$

Under the assumptions *H*1) — *H*5) the equality (3.1) defines a function $F : R \to R$, which is called the *optimal value function* of the parametrical problem $PA(t)$, $t \in R$.

Next we will present some useful properties of the optimal value function F. Firstly, we note that, by *H*1) — *H*5), we can consider the function $E : R \times X \to R$, where

$$E(t, x) = \min_{y \in T(x)} q(t, x, y) \text{ for all } (t, x) \in R \times X. \tag{3.3}$$

LEMMA 3.1. ([15]). *The function F is a nondecreasing function.*

LEMMA 3.2. *If the assumptions H*1) — *H*5) *hold, then F is a continuous function on R.*

Proof. From *H*4) and (3.2) it results that q is a continuous function. Then using the "Theorem of maximum" (see [3] or [18]) for problem

(3.3), it follows that E is a continuous function too. By applying again the "Theorem of maximum" to the problem

$$F(t) = \max_{x \in X} E(t, x), \tag{3.3'}$$

we get that F is continuous.

LEMMA 3.3. *If t is a real number such that there exists $x' \in X$ for which*

$$t = \min_{y \in T(x')} h(x', y), \tag{3.4}$$

then

$$F(t) \geqslant 0. \tag{3.5}$$

Proof. From (3.4) and (2.2) it results that

$$t \leqslant \frac{f(x', y)}{g(x', y)} \text{ for all } y \in T(x'),$$

whence, by $H3$), we get

$$f(x', y) - t\,g(x', y) \geqslant 0 \text{ for all } y \in T(x'),$$

which means that

$$E(t, x') = \min_{y \in T(x')} (f(x', y) - t\,g(x', y)) \geqslant 0. \tag{3.6}$$

But from (3.3′) it is evident that

$$F(t) \geqslant E(t, x').$$

Therefore, by (3.6), we can conclude that (3.5) holds.

LEMMA 3.4. *For a given real number t', let $x'' \in X$ be such that*

$$F(t') = E(t', x''), \tag{3.7}$$

and let $y'' \in T(x'')$ be an optimal solution for the minimization problem:

$$t'' = \min_{y \in T(x'')} h(x'', y). \tag{3.8}$$

Then the following inequality holds:

$$t'' - t' \geqslant \frac{F(t')}{g(x'', y'')}. \tag{3.9}$$

Proof. Indeed, by (3.7), (3.1) and (3.3), we have

$$F(t') = \min_{y \in T(x'')} (f(x'', y) - t'g(x'', y)) \leqslant f(x'', y'') - t'g(x'', y'').$$

But, using H3), it results that

$$h(x'', y'') - t' \geqslant \frac{F(t')}{g(x'', y'')},$$

whence, by (3.8), we deduce that (3.9) holds.

Next we will present some necessary and sufficient optimality conditions for the problem FMM.

THEOREM 3.1 ([15]). *Let $x' \in X$, $y' \in T(x')$ and $t' \in R$ be such that*

$$t' = h(x', y') = \min_{y \in T(x')} h(x', y). \tag{3.10}$$

Then (x', y') is an optimal solution for the problem FMM if and only if $F(t') = 0$.

THEOREM 3.2. *$F(t') = 0$ if and only if $V = t'$.*

Proof. Let us suppose that $V = t'$. Then by Lemma 2.2 there exists $(x', y') \in X \times Y$, which is an optimal solution for the problem FMM. Therefore, by Definition 2.1,

$$t' = \min_{y \in T(x')} h(x', y),$$

and then it follows from Theorem 3.1 that $F(t') = 0$.

Now, let us assume that $F(t') = 0$ and that $V \neq t'$. First, let us suppose that $t' < V$. Then, by Lemma 2 from [15], the equality $F(t') = 0$ implies $V \leqslant t'$. But this inequality contradicts the supposition that $t' < V$.

Let us suppose now that $V < t'$. Then there is a positive number r, such that $t' - r = V$.

Consequently, we have

$$t' - r \geqslant \min_{y \in T(x)} h(x, y), \text{ for all } x \in X. \tag{3.11}$$

For every $x \in X$, let $M(x)$ be the set of optimal solutions for the minimization problem (2.9), that is

$$M(x) = \{y'' \in T(x) \mid h(x, y'') = H(x)\}. \tag{3.12}$$

Therefore, from (3.11) we get

$$t' - r \geqslant \frac{f(x, y'')}{g(x, y'')} \text{ for all } x \in X \text{ and } y'' \in M(x),$$

whence, from $H3$), we obtain

$$f(x, y'') - t'g(x, y'') \leqslant -r\,g(x, y'') \text{ for all } x \in X \text{ and } y'' \in M(x). \tag{3.13}$$

But, from (3.13) it follows that

$$\min_{y \in T(x)} (f(x, y) - t'g(x, y)) \leqslant -rg(x, y''), \text{ for all } x \in X \text{ and } y'' \in M(x). \tag{3.14}$$

Let (x', y') be an optimal solution for the problem $PA(t)$. Then, from (3.14) we get at once that

$$F(t') \leqslant -rg(x', y') < 0,$$

which contradicts the assumption that $F(t') = 0$. Consequently, the supposition that $t' \neq V$ is not true. Therefore, we can conclude that $t' = V$.

From Theorem 3.2 and Lemma 3.1 we immediately obtain the following consequence.

Consequence 3.1. (*i*) $F(t) > 0$ *if and only if* $V > t$.
(*ii*) $F(t) < 0$ *if and only if* $V < t$.

Proof: (*i*) Let us prove that $V > t$ implies $F(t) > 0$. By Lemma 3.1, the inequality $V > t$ implies $F(t) \geq F(V) = 0$. If $F(t) = 0$, then by Theorem 3.2 it results that $t = V$, which contradicts the inequality $V > t$. Therefore, $F(t) > 0$.

Conversely, let us prove that $F(t) > 0$ implies $V > t$. Indeed, if we suppose that $V \leq t$, then by Lemma 3.1, it follows that

$$F(t) \leq F(V) = 0,$$

which contradicts the inequality $F(t) > 0$. Hence $V > t$.

The part (*ii*) is an obvious consequence of part (*i*) and Theorem 3.2.

THEOREM 3.3. *Let* $t' \in R$ *and* $x' \in X$ *be such that*

$$t' \leq V, \tag{3.15}$$

$$F(t') = E(t', x'),$$

and let $y' \in T(x')$ *be an optimal solution for the minimization problem*:

$$t'' = \min_{x \in T(x')} h(x', y).$$

If $t' = t''$, *then* (x', y') *is an optimal solution for the problem* FMM.

Proof. From Lemma 3.4 we have

$$t'' - t' \geq \frac{F(t')}{g(x', y')}. \tag{3.16}$$

But, from (3.15) and Consequence 3.1 it follows that

$$F(t') \geq 0. \tag{3.17}$$

On the other hand, by making $t'' = t'$ in (3.16), we get $F(t') \leq 0$, which together with (3.17) implies that $F(t') = 0 = F(t'')$.

Therefore, according to Theorem 3.1 and Definition 2.1, (x', y') is an optimal solution for the problem FMM.

The following theorem gives an upper bound of the optimal value V of the FMM problem.

THEOREM 3.4. *If the function* g *satisfies the condition*:

$$g(x, y) \geq \beta > 0 \text{ for all } x \in X \text{ and } y \in T(x), \tag{3.18}$$

then

$$V \leq t + \frac{F(t)}{\beta} \tag{3.19}$$

for every real number t.

Proof. Let (x', y') be an optimal solution of the problem FMM and let $y'' \in T(x')$ be such that

$$f(x', y'') - t\, g(x', y'') = \min_{y \in T(x')} (f(x', y) - t\, g(x', y)).$$

Then it follows by (3.1) that

$$f(x', y'') - t\, g(x', y'') \leq F(t),$$

whence, due to assumption $H3$), we get

$$h(x', y'') - t \leq \frac{F(t)}{g(x', y'')}. \tag{3.20}$$

But, by (2.6) we have

$$h(x', y') \leq h(x', y''),$$

and so it results that

$$h(x', y') - t \leq h(x', y'') - t \leq \frac{F(t)}{g(x', y'')}.$$

Then, taking into account (3.18), it follows that

$$h(x', y') - t \leq \frac{F(t)}{\beta},$$

whence, since from Definition 2.1 $V = h(x', y')$, we get (3.19).

4. The Parametrical Procedure. In this section firstly we recall the parametrical procedure for solving the fractional max-min problem FMM given in Ref. [15]. Also, using the optimality condition given in Theorem 3.3 as a stop criterion, we consider a modification of this procedure, which will be applied later (see Section 6) in the case of separable fractional max-min problems.

Each of these algorithms produces a sequence of points (x_k, y_k) in $X \times Y$ such that the sequence $(h(x_k, y_k))$ of the corresponding objective function values converges to the optimal value V of the fractional max-min problem FMM.

Algorithm 1

Step 1. Choose $x_0 \in X$ and set $k := 0$.
Step 2. Find $t_k \in R$ and $y_k \in T(x_k)$ such that

$$t_k = h(x_k, y_k) = \min_{y \in T(x_k)} h(x_k, y). \tag{4.1}$$

Step 3. Find an optimal solution (x_{k+1}, y') and the optimal value $F(t_k)$ of the auxiliary max-min problem $PA(t_k)$, that is

$$F(t_k) = \max_{x \in X} \min_{x \in T(x)} (f(x, y) - t_k\, g(x, y)). \tag{4.2}$$

Step 4. *i*) If $F(t_k) = 0$, then the algorithm stops, since according to Theorem 3.1, (x_k, y_k) is an optimal solution for the problem FMM.
ii) If

$$F(t_k) > 0 \tag{4.3}$$

then take $k := k + 1$ and go to Step 2.

The continuation criterion (4.3) in Step 4 is justified by Lemma 3.4. It yields an improvement of the objective function value in the next iteration, i.e.

(4.4) $$t_{k+1} = H(x_{k+1}) > H(x_k) = t_k.$$

Now, under the assumptions $H1) - H5)$, a result concerning the convergence of Algorithm 1 is given.

THEOREM 4.1. *If*

(4.5) $$F(t_k) > 0 \text{ for every natural number } k,$$

then:

i) $\lim_{k\to\infty} F(t_k) = 0$,

ii) $\lim_{k\to\infty} t_k = V$,

iii) the sequence $((x_k, y_k))$ *of the feasible solutions for the problem FMM generated by the parametrical procedure* (Algorithm 1) *has at least an accumulating point, and every accumulating point of this sequence is an optimal solution of the FMM problem.*

Proof. The conclusions *i)* and *ii)* of the theorem result by Theorem 4 from Ref. [15].

We will show now that there exists at least an accumulating point of the sequence $((x_k, y_k))$. Indeed, since X is a compact set (see, $H1$)), there is a convergent subsequence (x_{j_k}) of the sequence (x_k). Let us denote

(4.6) $$x' = \lim_{k\to\infty} x_{j_k}.$$

On the other hand, from the continuity of the point-to-set mapping T, it results that for every $r > 0$, there exists a natural number $K(r)$ such that

(4.7) $$T(x_{j_k}) \subseteq V_r(T(x')) \text{ for all } k \geqslant K(r),$$

where

(4.8) $$V_r(T(x')) = \{y \in R^m / \exists y' \in T(x') \text{ such that } \|y - y'\| \leqslant r\}.$$

Since $T(x')$ is by H3) a compact set, the set $V_r(T(x'))$ is a compact set too.

Therefore, by (4.7), all terms of the sequence (y_{j_k}) belongs to the compact set $V_r(T(x'))$ except a finite number. Then this sequence contains a convergent subsequence $(y_{j_{i_k}})$. Let us denote

(4.9) $$y' = \lim_{k\to\infty} y_{j_{i_k}};$$

(4.10) $$x'_k = x_{j_{i_k}} \text{ and } y'_k = y_{j_{i_k}}.$$

From (4.6) and (4.9) it results that (x', y') is an accumulating point for the sequence $((x_k, y_k))$.

Now let us show that (x', y') is an optimal solution of the FMM problem. The continuity of T and the assumption $H_1)$ imply the closeness of T (see [3]), whence from (4.6)and (4.9) we get that

(4.11) $$y' \in T(x').$$

Also, by the continuity of h, from (4.6), (4.9) and (4.10) it results that

(4.12) $$\lim_{k\to\infty} h(x'_k, y'_k) = h(x', y').$$

On the other hand, from *ii)* it follows that

$$\lim_{k\to\infty} h(x'_k, y'_k) = V,$$

whence, by (4.10), we obtain:

(4.13) $$h(x', y') = V.$$

Now, by Lemma 2.1, the function H (given by (2.9)) is a continuous function, so we have

(4.14) $$H(x') = \lim_{k\to\infty} H(x'_k),$$

whence, taking into account (4.1) and (2.9), it results that

(4.15) $$\lim_{k\to\infty} H(x'_k) = \lim_{k\to\infty} h(x'_k, y'_k).$$

Also, by (4.12) — (4.14), one obtains:

(4.16) $$h(x', y') = H(x') = \min_{y\in T(x')} h(x', y).$$

Hence according to Definition 2.1 it follows, from (4.6), (4.13) and (4.16) that (x', y') is an optimal solution of the FMM problem.

Now we present a variant of Algorithm 1 in which, by using the optimality condition from Theorem 3.3, we change the stop criterion in Step 4.

Algorithm 2

Initial phase:

Step 1. Choose $x_0 \in X$ and take $k := 0$.

Step 2. Find $t_0 \in R$ and $y_0 \in T(x_0)$ such that

$$t_0 = h(x_0, y_0) = \min_{y\in T(x_0)} h(x_0, y).$$

General phase:

Step 3. Find $x_{k+1} \in X$ for which there exists $y' \in T(x_{k+1})$ such that (x_{k+1}, y') is an optimal solution for the auxiliary max-min problem $PA(t_k)$.

Step 4. Find $t_{k+1} \in R$ and $y_{k+1} \in T(x_{k+1})$ such that

$$t_{k+1} = h(x_{k+1}, y_{k+1}) = \min_{y \in T(x_{k+1})} h(x_{k+1}, y).$$

Step 5. *i*) If $t_{k+1} = t_k$, then the algorithm stops. By Theorem 3.3, $t_k = V$ and (x_{k+1}, y_{k+1}) is an optimal solution for the problem *FMM*.

ii) If $t_{k+1} - t_k > 0$, then go to Step 3 with $k := k + 1$.

It can be easily seen that each of the Algorithms 1 and 2 yields the same sequence $((x_k, y_k))$ of feasible solutions of the *FMM* problem. However the algorithm 2 is more suitable than Algorithm 1 to be applied in the case of a separable fractional max-min problems, which will be considered later in Section 6.

In the next section we will give some approximate variants of Algorithm 1, which are more practical from the numerical point of view.

5. Approximate Finite Variants of the Parametrical Procedure. The parametrical procedure (Algorithm 1) considered in the previous section generally requires an infinite number of iterations. However, it can be stoped after a finite number of iterations by using an approximate stop criterion in Step 4. For instance, Step 4 in Algorithm 1 can be replaced be the following:

Step 4′. *i*) If $F(t_k) \leqslant \alpha$, then stop.

ii) If $F(t_k) > \alpha$, then take $k := k + 1$ and go to Step 2.

In Step 4′, α is a given nonnegative real number. To obtain a good approximation of the optimal solution, α must be taken sufficiently small.

Concerning this approximate variant of Algorithm 1 the following theorem may be stated.

THEOREM 5.1 *If the condition* (3.16) *holds and if* α *in* Step 4′, *is a positive real number, then the approximate variant of Algorithm* 1 *ends after a finite number of iterations with an* $\frac{\alpha}{\beta}$-*optimal solution for the FMM problem.*

Proof. Indeed, if we suppose that

$$F(t_k) > \alpha \text{ for any } k, \tag{5.1}$$

then, by Theorem 4.1, it follows that

$$\lim_{k \to \infty} F(t_k) = 0,$$

which contradicts (5.1).

Hence, there exists a natural number k such that

$$F(t_k) \leqslant \alpha.$$

Then, by Theorem 3.4, we have

$$V - t_k \leqslant \frac{F(t_k)}{\beta} \leqslant \frac{\alpha}{\beta}.$$

Hence

$$V - h(x_k, y_k) \leqslant \frac{\alpha}{\beta},$$

and by (4.1) it results that (x_k, y_k) is an $\frac{\alpha}{\beta}$-optimal solution for the *FMM* problem.

The Algorithm 1 (or 2) as well as its approximate variant needs in Step 2 the finding of an exact optimal solution for an ordinary nonlinear fractional programming problem (see (4.1)). Also, in Step 3 a max-min problem (see 4.2)) must be solved. But for these optimization problems, especially in the nonlinear cases, only approximate solutions can be obtained.

Therefore it is useful to consider some variants of the algorithms 1 or 2 that need in the steps 2 and 3 only approximate solutions with a prescribed approximation.

Next we will consider an approximate variant of Algorithm 1, which needs at every iteration a γ-optimal solution for the fractional programming problem (4.1) in Step 2 and a δ-optimal solution for the max-min problem (4.2) in Step 3. Here γ and δ are given nonnegative real numbers, which represent the desired approximations in the Steps 2 and 3 respectively.

Algorithm 3

Step 1. Choose $x_0 \in X$ and take $k := 0$.

Step 2. Find $y_k \in T(x_k)$ such that

$$h(x_k, y_k) - \gamma \leqslant \min_{y \in T(x_k)} h(x_k, y) = H(x_k) \tag{5.2}$$

and take

$$t_k = h(x_k, y_k). \tag{5.3}$$

Step 3. Find $x_{k+1} \in X$ for which there exists $y' \in T(x_{k+1})$ such that (x_{k+1}, y') is a δ-optimal solution for the max-min problem $PA(t_k)$, that is:

$$q(t_k, x_{k+1}, y') + \delta \geqslant F(t_k), \tag{5.4}$$

$$q(t_k, x_{k+1}, y') - \delta \leqslant \min_{y \in T(x_{k+1})} q(t_k, x_{k+1}, y). \tag{5.5}$$

Let us denote

$$B_k = q(t_k, x_{k+1}, y'). \tag{5.6}$$

Step 4. *i)* If $B_k > \alpha$, then go to Step 2 with $k := k + 1$.

ii) If $B_k \leqslant \alpha$, then the algorithm stops.

Next we will derive a sufficient condition for the finite convergence of the approximate Algorithm 3.

Firstly we obtain an upper bound of the optimal value V of the problem FMM.

THEOREM 5.2. *If the condition* (3.16) *holds and if*

$$B_k \leqslant \alpha, \tag{5.7}$$

then

$$V \leqslant t_k + \frac{\alpha + \delta}{\beta}, \tag{5.8}$$

and (x_k, y_k) *is an* ε*-optimal solution for the* FMM *problem, where*

$$\varepsilon = \max\left(\gamma, \frac{\alpha + \delta}{\beta}\right).$$

Proof. From (5.4) and (5.6) we have

$$F(t_k) \leqslant B_k + \delta,$$

whence, by (5.7), it follows that

$$F(t_k) \leqslant \alpha + \delta. \tag{5.9}$$

On the other hand, by Theorem 3.4 we have

$$V \leqslant t_k + \frac{F(t_k)}{\beta}. \tag{5.10}$$

But (5.9) and (5.10) imply (5.8).

In order to prove that (x_k, y_k) is an ε-optimal solution of the FMM problem, we make the remark that in Step 2 of the Algorithm 3, y_k is determined in $T(x_k)$ such that

$$h(x_k, y_k) - \gamma \leqslant \min_{y \in T(x_k)} h(x_k, y). \tag{5.11}$$

From (5.8) and (5.11) we may conclude that (x_k, y_k) satisfies the conditions (2.4), (2.7) and (2.8) of Definition 2.2 with

$$\varepsilon = \max\left(\gamma, \frac{\alpha + \delta}{\beta}\right).$$

Hence (x_k, y_k) is an ε-optimal solution for the FMM problem.

LEMMA 5.2. *If* $B_k > \alpha$, *then*

$$t_{k+1} - t_k > \frac{\alpha - \delta}{g(x_{k+1}, y_{k+1})}. \tag{5.12}$$

Proof. By (5.5) and (5.6) we have

$$B_k - \delta \leqslant \min_{y \in T(x_{k+1})} q(t_k, x_{k+1}, y) \leqslant q(t_k, x_{k+1}, y_{k+1}) =$$

$$= f(x_{k+1}, y_{k+1}) - t_k\, g(x_{k+1}, y_{k+1}),$$

whence, by $H3$), (2.2) and (5.3), it follows that

$$\frac{B_k - \delta}{g(x_{k+1}, y_{k+1})} \leqslant t_{k+1} - t_k. \tag{5.13}$$

Since $B_k > \alpha$, inequality (5.12) results immediately from (5.13).

THEOREM 5.3. *If in* Algorithm 3

$$\delta < \alpha \tag{5.14}$$

and if

$$0 < \beta \leqslant g(x, y) \leqslant \omega \text{ for all } x \in X \text{ and } y \in T(x), \tag{5.15}$$

then Algorithm 3 *finishes after a finite number of iterations with an* ε*-optimal solution of the* FMM *problem, where*

$$\varepsilon = \max\left(\gamma, \frac{\alpha + \delta}{\beta}\right).$$

Proof. Indeed, whenever $B_k > \alpha$, by (5.2) and (5.3) we obtain

$$t_k - \gamma \leqslant \min_{y \in T(x_k)} h(x_k, y) \leqslant V,$$

that is

$$t_k \leqslant V + \gamma. \tag{5.16}$$

On the other hand, from Lemma 5.2, whenever $B_k > \alpha$, we have

$$t_{k+1} - t_k > \frac{\alpha - \delta}{g(x_{k+1}, y_{k+1})},$$

whence, by (5.14) and (5.15) it follows

$$t_{k+1} - t_k > \frac{\alpha - \delta}{\omega} > 0. \tag{5.17}$$

But the inequalities (5.16) and (5.17) imply that (t_n) must be a finite sequence. Therefore, there exists a k' such that $B_{k'} \leqslant \alpha$. Then the Algorithm 3 stops and by Theorem 5.2, the pair (x_k, y_k) is an ε-optimal solution. This concludes the proof.

Remark. The assumption (5.15) in Theorem 5.3 is satisfied when, for instance, we assume toghether with $H1$) — $H5$) that Y is a compact set.

6. Separable Fractional MAX-MIN Problems. In this section we will present an adaptation of Algorithm 2 for the particular case of the fractional max-min problem FMM, when:

(6.1) $f(x, y) = M(x) + P(y)$, for all $(x, y) \in X \times Y$,

(6.2) $g(x, y) = N(x) + Q(y)$, for all $(x, y) \in X \times Y$,

(6.3) $T(x) = Y$, for all $x \in X$,

where $M : X \to R$, $N : X \to R$, $P : Y \to R$, $Q : Y \to R$ are given functions.

This max-min problem can be stated as follows:

SFM Find

$$V = \max_{x \in X} \min_{y \in Y} \frac{M(x) + P(y)}{N(x) + Q(y)}. \tag{6.4}$$

We call the problem *SFM* separable fractional max-min problem.

If we want to apply the Algorithm 2 to the max-min problem *SFM*, at the k-th iteration the following max-min problem in Step 3 must be solved:

$$F(t_k) = \max_{x \in X} \min_{y \in Y} (M(x) + P(y) - t_k(N(x) + Q(y))). \tag{6.5}$$

As a matter of fact, it must found only an $x_{k+1} \in X$ for which there exists $y' \in Y$ such that (x_{k+1}, x') is an optimal solution for the max-min problem (6.5), because the second part y' of the pair (x_{k+1}, y') is not used in the next iterations. Moreover, since the problem (6.5) can be written as

$$F(t_k) = \max_{x \in X} (M(x) - t_k N(x) + \min_{y \in Y} (P(y) - t_k Q(y)) =$$

$$= \max_{x \in X} (M(x) - t_k N(x)) + \min_{y \in Y} (P(y) - t_k Q(y)),$$

for finding x_{k+1} it is enough to solve the following usual maximization problem:

$PM(t_k)$. Find $x_{k+1} \in X$ such that

$$M(x_{k+1}) - t_k N(x_{k+1}) = \max_{x \in X} (M(x) - t_k N(x)). \tag{5.6}$$

We consider now for the problem *SFM* a variant of the algorithm 2, where the max-min problem $PA(t_k)$ is replaced by the ordinary maximization problem $PM(t_k)$.

Algorithm 4

Step 1. Choose $x_0 \in X$ and take $k := 0$.

Step 2. Find an optimal solution $y_k \in Y$ for the fractional minimization problem

$$t_k = \min_{y \in Y} \frac{M(x_k) + P(y)}{N(x_k) + Q(y)}. \tag{6.7}$$

Step 3. If x_k is an optimal solution for the problem $PM(t_k)$, then stop. Otherwise, go to the following step.

Step 4. Find an optimal solution $x_{k+1} \in X$ for the problem $PM(t_k)$ and go to Step 2 with $k := k + 1$.

The Algorithm 4 is effective especially when some supplementary convexity conditions on the *SFM* problem are imposed. For instance, if M is a concave function, N is a convex function, h is nonnegative over $X \times Y$ and X is a convex set, then the problem $PM(t_k)$ (in Step 4) is a concave programming problem, for which there exist efficient algorithms (see, for instance [1]). Also, the fractional minimization problem (6.7) (in Step 2) can be reduced under suitable hypotheses to convex programming problems (see, e.g. [11]).

7. Bilinear Fractional MAX-MIN Problems. The bilinear fractional max-min problem, which will be considered in this section, is the particular case of the max-min problem FMM, where the functions f and g are bilinear functions of the form:

(7.1) $f(x, y) = xA^1y + d^1x + a^1y + w^1$ for all $(x, y) \in X \times Y$,

(7.2) $g(x, y) = xA^2y + d^2x + a^2y + w^2$ for all $(x, y) \in X \times Y$,

and X and Y are polyhedral sets:

$$X = \{x \in R^n / Bx \leqslant b,\ x \geqslant 0\}, \tag{7.3}$$

$$Y = \{y \in R^m / Ey \geqslant e,\ y \geqslant 0\}. \tag{7.4}$$

In (7.1) – (7.4) x and y are variable vectors, whereas $A^i \in R^{m \times n}$ $(i = 1, 2)$, $B \in R^{m \times p}$, $E \in R^{n \times s}$, $d^i \in R^n$, $a^i \in R^m$ $(i = 1, 2)$, $b \in R^p$, $e \in R^s$, $w^i \in R$ $(i = 1, 2)$ are given matrices, vectors and constants, respectively.

Therefore, the bilinear fractional max-min problem can be stated as follows:

BFM. Find

$$V = \max_{x \in X} \min_{y \in Y} \frac{xA^1 y + d^1x + a^1y + w^1}{xA^2 y + d^2x + a^2y + w^2} \tag{7.5}$$

The particular case of the problem *BFM* when $A^2 = 0$ and $a^2 = 0$, was considered by Belenkii [2].

Such problems arrise also from the minimum risk approach applied to the max-min bilinear programming problem (see, [13]).

Throughout this section we suppose that X and Y are nonempty and compact sets and that g satisfies the condition $H3$).

Concerning the use of Algorithm 1 to solve the problem *BFM* we can make the following remarks

Remark 7.1. In Step 2 only an usual linear, fractional programming problem must be solved. Moreover, from one iteration to other, it changes only the objective function of this fractional problem (see (4.1)). So, if a simplex type algorithm is used to solve it, then at $k + 1$-th iteration the optimal solution y_k obtained in the previous iteration can be taken as initial solution.

Remark 7.2 In Step 3, instead of the auxiliary max-min problem $PA(t_k)$ only an usual linear programming problem (see, e.g. [9]) must be solved.

Indeed, taking into consideration the relations (7.1)–(7.5), the nonfractional max-min problem $PA(t_k)$, which must be solved in Step 3 is the following:

(7.6) $$F(t_k) = \max_{x\in X} \min_{y\in Y} (x(A^1 - t_k A^2) y + (d^1 - t_k d^2) x + (a^1 - t_k a^2) y + (w^1 - t_k w^2)).$$

But, for a fixed element x in X, we get from (7.6) the linear programming problem:

(7.7) $$E(t_k, x) = \min_y ((x(A^1 - t_k A^2)+a^1-t_k a^2) y+(d^1-t_k d^2) x+w^1-t_k w^2)$$

subject to:

(7.8) $$Ey \geqslant e,$$

(7.9) $$y \geqslant 0.$$

Then, by the duality property of linear programming, we have for every $x \in X$:

(7.10) $$E(t_k, x) = \max_v (ev + (d^1 - t_k d^2) x + w^1 - t_k w^2)$$

subject to:

(7.11) $$vE \leqslant x(A^1 - t_k A^2) + a^1 - t_k a^2,$$

(7.12) $$v \geqslant 0,$$

where $v \in R^s$ is the vector of dual variables.

Taking into account the relation (3.3′), it is easily seen that for the finding of $F(t_k)$ the following linear programming problem must be solved: Find

(7.13) $$F(t_k) = \max_{x,v} (ev + (d^1 - t_k d^2) x + w^1 - t_k w^2)$$

subject to (7.11), (7.12) and

(7.14) $$Bx \leqslant b,$$

(7.15) $$x \geqslant 0.$$

Also, if (x', v') is an optimal solution of the problem (7.13) – (7.15), then $x_{k+1} = x'$ is the element in X which must be determined in Step 3 of the Algorithm 1. So, it is possible to perform the Step 3 only by solving an usual linear programming problem.

Next let us perform in the bilinear fractional max-min problem *BFM* the Charnes-Cooper variable change:

(7.16) $$u = \theta x,\ z = \theta y,$$

where $u \in R^n$, $z \in R^m$ and $\theta \in R$. Then problem *BFM* becomes: *BL*. Find

(7.17) $$V = \max_u \min_{z,\theta} (uA^1 z + \theta d^1 u + \theta a^1 z + w^1 \theta^2)$$

subject to

(7.18) $$Bu - b\theta \leqslant 0,$$

(7.19) $$Ez - e\theta \geqslant 0,$$

(7.20) $$uA^2 z + \theta d^2 u + \theta a^2 z + w^2 \theta^2 = 1,$$

(7.21) $$u \geqslant 0,\ z \geqslant 0,\ \theta \geqslant 0.$$

Concerning the problems *BFM* and *BL* we can state the following result.

THEOREM 7.1. *Let X and Y be nonvoid compact sets and let g be a positive function on $X \times Y$. If (u', z', θ') is an optimal solution for the problem BL, then $\left(\frac{u'}{\theta'}, \frac{z'}{\theta'}\right)$ is an optimal solution for the BFM problem. Conversely, if (x', y') is an optimal solution for the BFM problem, then $(\theta' x', \theta' z', \theta')$ is an optimal solution for the problem BL, where*

$$\theta' = \frac{1}{g(x', y')}.$$

Proof. To prove the first part of the theorem, we observe that for every feasible solution (u, z, θ) of the max-min problem *BL* we have $\theta > 0$. In the contrar case, if we take $\theta = 0$ in the constraints (7.18) — (7.21) of the max-min problem *BL*, we obtain the following relations:

(7.22) $$Bu \leqslant 0,$$

(7.23) $$Ez \geqslant 0,$$

(7.24) $$uA^2 z = 1.$$

Then, because by (7.21) $u \geqslant 0$ and $v \geqslant 0$, it follows from (7.24), that (u, z) is not the null vector. But, taking into account the inequalities (7.22) and (7.23), if $x \in X$ and $y \in Y$, then, by (7.3) and (7.4), it results that

$$x + tu \in X,\ y + tz \in Y, \text{ for all } t \geqslant 0,$$

which is contrary to the assumption that the sets X and Y are compact. Now, by employing Theorem 1 from Ref. [17], one obtains the first part of the theorem.

The second part of the theorem can be easily obtained by performing in the *BFM* problem the Charnes-Cooper variable change (7.16).

Hence we observe that by Theorem 7.1, the max-min problem *BL* having a nonlinear quadratic objective function and nonlinear constraints, can be solved by using the parametrical Algorithm 1 (for the max-min problem *BFM*), i.e. by solving only linear programming problems (in Step 3) and linear-fractional programming problems (in Step 2).

Now we consider an example of a fractional max-min problem with linear separate constraints, for which the variable change method seems to be very efficient. Thus we will show that this problem can be solved by linear programming.

The fractional max-min problem that we deal with is :

P. Find

$$\max_{x\in X}\ \min_{y\in Y}\left(\frac{xAy+dx+ay+w}{\min\limits_{i\in K}(h^i y+r^i)}+cx\right),$$

where X and Y are polyhedral sets given by (7.3) and (7.4), whereas $A\in R^{m\times n}$, $d\in R^n$, $a\in R^m$, $h^i\in R^m$, $r^i\in R(i\in K=\{1,\dots,k\})$ $w\in R$ are given matrices, vectors and constants respectively.

Next we suppose that

$$G(y)=\min_{i\in K}(h^i y+r^i)>0,\ \forall y\in Y.$$

Now let us perform in the problem P the variable change :

$$z=\theta y,$$

where $z\in R^m$ and $\theta\in R$, $\theta\geqslant 0$. Then we get the problem :

P1. Find

$$\max_{x\in X}\ \min_{z,\theta}\ (xAz+\theta dx+az+w\theta+cx) \tag{7.25}$$

where z and θ are subjected to :

$$Ez-e\theta\geqslant 0, \tag{7.26}$$

$$\min_{i\in K}\ (h^i y+r^i\theta)\geqslant 1, \tag{7.27}$$

$$z\geqslant 0,\ \ \theta\geqslant 0. \tag{7.28}$$

Concerning the problems P and $P1$ we can state the following result.

THEOREM 7.2. *Let X and Y be nonvoid compact sets and let G be a positive function on Y. If (x', z', θ') is an optimal solution for the problem $P1$, then $\left(x', \frac{z'}{\theta'}\right)$ is an optimal solution for the problem P.*

Proof. The proof can be easily obtained by using similar arguments as in the proof of Theorem 7.1 (see also Refs. [5], [8] and [11]).

We remark that the problem $P1$ is equivalent to a bilinear max-min problem with linear constraints. This fact results since the nonlinear constraint (7.27) is equivalent to the system of linear constraints

$$h^i z+r^i\theta\geqslant 1,\ i\in K. \tag{7.29}$$

Let $P2$ be the max-min problem obtained from $P1$ by replacing the constraint (7.27) with (7.29). $P2$ is a bilinear max-min problem with separate linear constraints and so it can be solved (see [9]) by linear programming.

R E F E R E N C E S

[1]. Bazaraa M. S., Shetty C. M., *Nonlinear programming. Theory and Algorithms*, John Wiley and Sons, New-York, 1979.
[2]. Belenkii A. S., *Minimax scheduling problems with linear constraints and methods of their solution*, Avtom. i Telemeh., **10**(1981), 157–170.
[3]. Berge C., *Espaces topologiques*, Dunod, Paris, 1959.
[4]. Bitran G., Magnanti T. L., *Duality and sensitivity analysis for Fractional Programming*, Operations Res., **24**(1976), 4, 675–699.
[5]. Charnes A., Cooper W. W., *Programming with linear fractional functionals*, Nav. Res. Log. Quart., **9**(1962) 181–186.
[6]. Cook W. D., Kirby M. J. L., Mehndiratta S. L., *A linear fractional Max-min Problem*, Operations Res., **23**(1975), 3, 511–521.
[7]. Dinkelbach W., *On nonlinear fractional programming*, Manag. Sci., **13**(1967), 7, 492–498.
[8]. Mond B., Craven B. D., *Nonlinear fractional programming*, Bull. Austral. Math. Soc., **12**(1975), 391–397.
[9]. Owen G., *Game Theory*, W.B. Saunders Company, Philadelphia, 1963.
[10]. Peteanu V., Tigan S., *On some discrete fractional max-min problems*. Application to max-min problems in graphs, Analyse numér. théor. approx., **13**(1984), 2, 167–173.
[11]. Schaible S., *Nonlinear fractional programming*, Operations Res. Verfahren, **19**(1974), 109–115.
[12]. Schaible S., *Fractional programming with several ratios*, Methods of Operations Research, **49**(1985), 77–83.
[13]. Stancu-Minasian I. M., Tigan S., *The minimum risk approach to max-min bilinear programming*, Analele Univ. „A. I. Cuza" Iași, Tom XXXI, 5 II b, (1985), 205–209.
[14]. Tigan S., *On a method for fractional optimization problems. Application to stochastic optimization problems*, Proc. of the Computer Science Conference, Székesfehérvar, Hungary (1973), 351–355.
[15]. Tigan S., *A parametrical method for max-min nonlinear fractional problems*, Itinerant Seminar on functional equations, approximation and convexity, Cluj-Napoca, (1983), 175–184.
[16]. Tigan S., *Sur une méthode pour la résolution d'un problème d'optimisation fractionnaire par segments*, Mathematica — Revue d'analyse numérique et de théorie de l'approximation, **4**(1975), 1, 87–97.
[17]. Tigan S., *On the max-min nonlinear fractional problem*, Mathematica — Revue d'analyse numérique et de théorie de l'approximation, **9**(1980), 2, 283–288.
[18]. Tigan S., *Sur quelques propriétés de stabilité concernant les problèmes d'optimisation avec contraintes*, Mathematica — Revue d'analyse numérique et de théorie de l'approximation, **6**(1977), 2, 203–225.
[19]. Tigan S., Stancu-Minasian I. M., *The stochastic Max-min problem*, Operations Res. Verfahren, **51**(1984), 119–126.
[20]. Tigan S., *On some nonlinear knapsack problems*, Itinerant Seminar on functional equations, approximation and convexity, Babeș-Bolyai Univ. Cluj-Napoca, 1984, 203–208.

Received 11.IX.1987

Centrul Teritorial de Calcul Electronic
Str. Republicii, Nr. 107
3400 Cluj-Napoca
România