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A KOROVKIN-TYPE THEOREM FOR GENERALIZATIONS OF BOOLEAN SUM OPERATORS AND APPROXIMATION BY TRIGONOMETRIC PSEUDOPOLYNOMIALS

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Abstract. We prove a Korovkin-type theorem on approximation via generalizations of Boolean sum operators in a space of B-continuous functions which are periodic in a certain sense. As an application we treat the problem of uniformly approximating B-continuous functions by trigonometric pseudopolynomials.

1. Introduction. In recent years there has been some interest in the approximation of bivariate functions by Boolean sums of parametric extensions of univariate approximation operators, especially in connection with problems in the field of Computer Aided Geometric Design; cf. e.g. the corresponding sections in [8] and the references cited there, as well as the discussion in the recent paper [1]. If the underlying univariate operators define (algebraic or trigonometric) polynomial approximants the corresponding bivariate approximants are algebraic or trigonometric *pseudopolynomials*, i.e. of the form

$$\sum_{i=0}^{m} A_i(y)\cdot x^i + \sum_{j=0}^{n} B_j(x)\cdot y^j,$$

$$\text{(1.1)}\qquad \text{or } \sum_{i=0}^{2m} A_i(y)\cdot \tau_i(x) + \sum_{j=0}^{2n} B_j(x)\cdot \tau_j(y),$$

$$\tau_i(z) = \begin{cases} \sin\dfrac{i+1}{2}z & \text{if } i \text{ is odd} \\ \cos\dfrac{i}{2}z & \text{if } i \text{ is even} \end{cases},$$

where A_i, B_j are univariate coefficient functions which should be periodic in the trigonometric case.

In most investigations the approximated functions are assumed to be continuous. However, the considered approximation processes are often meaningful for a bigger class of functions, namely for so-called B-continuous functions introduced by K. Bögel in [4] (cf. also [5], [6]).

A real valued function f on $D \subset \mathbb{R}^2$ is called *B-continuous*, if for every $(x, y) \in D$ we have

$$\lim_{(u,v)\to(x,y)} \Delta_{u,v}f(x, y) = 0,$$

where $\Delta_{u,v}f(x, y) := f(x, y) - f(x, v) - f(u, y) + f(u, v)$.

In [2] a Korovkin-type theorem on approximation in the space $B(I^2)$ of B-continuous functions defined on the unit square I^2 with certain generalizations of Boolean sum operators was proved.

In this paper we give an analogue for the approximation of certain "periodic" functions satisfying

$$\Delta_{u,v}f(x + 2\pi, y + 2\pi) = \Delta_{u,v}f(x, y) \tag{1.2}$$

for each $(x, y), (u, v) \in \mathbb{R}^2$.

Such functions we call $B - 2\pi$-periodic, and we denote by $B_{2\pi}$ the space of all real-valued $B - 2\pi$-periodic and B-continuous functions on $\mathbb{R}^2$.

Before turning to our Korovkin-type theorem for $B_{2\pi}$ in Section 3 we give some results about the structure of this space in Section 2. In Section 4 as an application the problem of uniformly approximating elements of $B_{2\pi}$ by trigonometric pseudopolynomials is considered.

2. Some properties of $B-2\pi$-periodic functions. It is clear that $B_{2\pi}$ is a real vector space with the usual pointwise definition of addition and scalar multiplication. However, $B_{2\pi}$ is not closed under function multiplication. The product of two $B_{2\pi}$—functions need neither be B-continuous nor $B_{2\pi}$-periodic as can be seen by considering functions of the type $f(x, y) = g(x) + h(y)$ which are always in $B_{2\pi}$. Such functions we will call *B-constants* as they are "constant" pseudopolynomials with $m = n = 0$ in (1.1).

Also, $f \in B_{2\pi}$ does not imply $|f| \in B_{2\pi}$. Turning to the absolute value, again neither preserves B-continuity (consider an obvious modification of a corresponding example for the $B(I^2)$-case in [1]), nor B-2π-periodicity as shows the following

Example 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x, y) = (x - 2k_x \pi) \sin y + 2k_x\pi, \tag{2.1}$$

where $k_x \in \mathbb{Z}$ such that $x \in [2k_x\pi, 2(k_x + 1)\pi[$. We have

$$f(x, y) - f(x, v) = (x - 2k_x\pi)(\sin y - \sin v) =$$
$$= (x+2\pi-2(k_x+1)\pi)(\sin y - \sin v) = f(x + 2\pi, y) - f(x + 2\pi, v)$$

for $(x, y), (u, v) \in \mathbb{R}^2$. Since f is 2π-periodic with respect to the second variable it follows that

$$\Delta_{u,v} f(x, y) = \Delta_{u,v} f(x + 2\pi, y + 2\pi)$$

for all $(x, y), (u, v) \in \mathbb{R}^2$.

Furthermore

$$|\Delta_{u,v}f(x, y)| = |x - u + 2(k_u - k_x)\pi| \cdot |\sin y - \sin v| \to 0 \quad \text{for} \quad (u, v) \to (x, y);$$

i.e. f is B-continuous. (One can even check that $|f|$ is B-continuous.) Thus $f \in B_{2\pi}$. But we have for example

$$\left|f\left(-\pi, \frac{\pi}{2}\right)\right| - |f(-\pi, 0)| = -\pi$$

and

$$\left|f\left(\pi, \frac{\pi}{2}\right)\right| - |f(\pi, 0)| = \pi,$$

from which it follows that

$$\Delta_{u,0}|f|\left(-\pi, \frac{\pi}{2}\right) \neq \Delta_{u,0}|f|\left(\pi, \frac{5\pi}{2}\right),$$

i.e. $|f|$ is not $B_{2\pi}$-periodic. ∎

The space $C_{2\pi,2\pi}$ of all bivariate continuous and real-valued functions which are 2π-periodic with respect to both variables is a proper subspace of $B_{2\pi}$ as can be shown again via the example of B-constants or via the preceeding example: A $B_{2\pi}$-function in general neither satisfies the continuity nor the periodicity properties of $C_{2\pi,2\pi}$-functions. It even need not be bounded.

The situation is different for the associated difference function $\Delta_{u,v}f$ with fixed $(u, v) \in \mathbb{R}^2$. For $f \in B_{2\pi}$ this function is bounded and 2π-periodic with respect to both variables as the following two lemmas will show.

LEMMA 2.2. *If f is $B - 2\pi$-periodic, then*

$$\Delta_{u+2h\pi,v+2k\pi}f(x + 2m\pi, y + 2n\pi) = \Delta_{u,v}f(x, y) \tag{2.2}$$

for every $(x, y), (u, v) \in \mathbb{R}^2$, where h, k, m, and n are integers.

Proof. There are six steps in the proof.

(a) We prove (2.2) when $h = k = 0$, $m = 1$, and n is a positive integer. This is done by mathematical induction.

If $n = 1$, (2.2) is true by hypothesis. Now we assume that

$$\Delta_{u,v} f(x + 2\pi, y + 2n\pi) = \Delta_{u,v} f(x, y)$$

for every $(x, y), (u, v) \in \mathbb{R}^2$. Moreover, we have

$$\Delta_{u,y+2n\pi} f(x + 2\pi, y + 2n\pi + 2\pi) = \Delta_{u,y+2n\pi} f(x, y + 2n\pi) = 0,$$

and adding these two equalities we find

$$\Delta_{u,v} f(x + 2\pi, y + (2n + 2)\pi) = \Delta_{u,v} f(x, y),$$

i.e. the desired result.

(b) In a similar manner we can prove, using step (a), that (2.2) is true for $h = k = 0$, and positive integers m, n.

(c) Now we prove (2.2) when $h = k = 0$, $m = 0$, and n is a positive integer.

We have

$$\Delta_{u,v} f(x, y + 2n\pi) = \Delta_{u,v} f(x, y) + \Delta_{u,v} f(x, y + 2n\pi).$$

But, using again (a), we may write

$$\Delta_{u,y} f(x, y + 2n\pi) = \Delta_{u,y} f(x - 2\pi + 2\pi, y + 2n\pi) = \Delta_{u,y} f(x - 2\pi, y) = 0.$$

(d) Similarly, we can prove (2.2) for $h = k = 0$, $l = 0$, and m a positive integer.

Hence, in these four steps we proved (2.2) for $h = k = 0$, and m, n non-negative integers.

(e) Now we prove (2.2) when h, k, m, n are non-negative integers. Indeed, using the above steps, we have

$$\Delta_{u+2h\pi, v+2k\pi} f(x + 2m\pi, y + 2n\pi) = \Delta_{u+2h\pi, v+2k\pi} f(x, y)$$

$$= \Delta_{x,y} f(u + 2h\pi, v + 2k\pi) = \Delta_{x,y} f(u, v) = \Delta_{u,v} f(x, y).$$

(f) The equality (2.2) is also true if some of the integers h, k, m, n are negative. This follows easily from (e) because the last equalities are valid for every $(x, y),(u, v) \in \mathbb{R}^2$.

Thus the proof is complete. ∎

Lemma 2.2 is a generalization of the periodicity property of $\Delta_{u,v} f$ stated above. The boundedness result which was given in [3, Lemma] for the $B(I^2)$-case we formulate in

LEMMA 2.3. *If f is an element of the space $B_{2\pi}$, then there is a positive number $M = M(f)$ such that for every (x, y) and (s, t) from $\mathbb{R}^2$ we have*

$$|\Delta_{s,t} f(x, y)| \leqslant M.$$

Proof. Any real number u can be written in the form

$$u = u_1 + 2\pi u_2$$

where $u_1 \in [0, 2\pi[$, and u_2 is an integer $\left(\text{the integral part of } \dfrac{u}{2\pi}\right)$. Using this decomposition for x, y, s, and t, and Lemma 2.2 we obtain

$$(2.3) \quad \Delta_{s,t} f(x, y) = \Delta_{s_1+2\pi s_2, t_1+2\pi t_2} f(x_1 + 2\pi x_2, y_1 + 2\pi y_2) = \Delta_{s_1,t_1} f(x_1, y_1).$$

Now, Lemma 2.3 follows from (2.3) and from [3, Lema]. ∎

COROLLARY 2.4. *Each $f \in B_{2\pi}$ may be written as a sum of a bounded B-continuous function which is 2π-periodic with respect to each variable and a B-constant.*

Proof. We may write

$$f(x, y) = \Delta_{x_0,y_0} f(x, y) + f(x_0, y) + f(x, y_0) - f(x_0, y_0)$$

with fixed $(x_0, y_0) \in \mathbb{R}^2$.

According to Lemma 2.2 and Lemma 2.3 the function $\Delta_{x_0,y_0} f$ is 2π-periodic with respect to each variable and bounded. $\Delta_{x_0,y_0} f$ is also B-continuous since $\Delta_{u,v}(\Delta_{x_0,y_0} f)(x, y) = \Delta_{u,v} f(x, y)$. ∎

Remark. It is not true that $f \in B_{2\pi}$ can be written as a sum of a $C_{2\pi,2\pi}$-function and a B-constant. Compare the example of a B-continuous function which is not continuous up to a B-constant in [6].

The preceeding considerations enable us to show

LEMMA 2.5. *Each $f \in B_{2\pi}$ is uniformly B-continuous on $\mathbb{R}^2$; i.e. for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for every $(x, y), (u, v) \in \mathbb{R}^2$ with $|x - u| \leqslant \delta(\varepsilon)$ and $|y - v| \leqslant \delta(\varepsilon)$ we have*

$$(2.4) \qquad |\Delta_{u,v} f(x, y)| \leqslant \varepsilon.$$

Proof. Because f is B-continuous on $\mathbb{R}^2$ it follows that f is also B-continuous on $[0, 3\pi]^2 = [0, 3\pi] \times [0, 3\pi]$, and then by [6, Satz 7] the function f is uniformly B-continuous on $[0, 3\pi]^2$. Thus there exists a function $\delta_1(\varepsilon) > 0$, $\varepsilon > 0$, such that (2.4) is satisfied for every $(x, y),(u, v) \in [0,3\pi]^2$ with $|x - u| \leqslant \delta_1(\varepsilon)$ and $|y - v| \leqslant \delta_1(\varepsilon)$.

We now define the function $\delta(\varepsilon)$ from the definition of uniform B-continuity of f on $\mathbb{R}^2$ by $\delta(\varepsilon) = \min\{\pi, \delta_1(\varepsilon)\}$, and show that (2.4) holds for every $(x, y),(u, v) \in \mathbb{R}^2$ with $|x - u| \leqslant \delta(\varepsilon)$ and $|y - v| \leqslant \delta(\varepsilon)$. Because $\delta(\varepsilon) \leqslant \pi$, we can choose $k, l \in \mathbb{Z}$ such that

$$x = x_1 + 2k\pi,\ x_1 \in [0, 3\pi];$$

$$u = u_1 + 2k\pi,\ u_1 \in [0, 3\pi];$$

$$y = y_1 + 2l\pi,\ y_1 \in [0, 3\pi];$$

$$v = v_1 + 2l\pi,\ v_1 \in [0, 3\pi].$$

Then

$$|x_1 - u_1| = |x - u| \leqslant \delta(\varepsilon) \leqslant \delta_1(\varepsilon) \text{ and } |y_1 - v_1| = |y - v| \leqslant \leqslant \delta(\varepsilon) \leqslant \delta_1(\varepsilon).$$

Because $(x_1, y_1),(u_1, v_1) \in [0, 3\pi]^2$ and f is uniformly B-continuous on $[0, 3\pi]^2$ we obtain $|\Delta_{u_1,v_1} f(x_1, y_1)| \leqslant \varepsilon$. Using this inequality and Lemma 2.2 we get

$$|\Delta_{u,v} f(x, y)| = |\Delta_{u_1,v_1} f(x_1, y_1)| \leqslant \varepsilon.$$

Now the proof is complete. ∎

3. A Korovkin-type theorem for approximation in $B_{2\pi}$ with generalizations of Boolean sum operators.

Having supplied the necessary auxiliary results in Section 2 we may now proceed similarly as in the proof of the Korovkin-type theorem in [2] to obtain our main assertion.

We first prove the following

LEMMA 3.1. *Let $f \in B_{2\pi}$ be arbitrarily chosen. For every positive number ε there are two positive numbers $A(\varepsilon) = A(\varepsilon, f)$ and $B(\varepsilon) = B(\varepsilon, f)$ such that for every $(x, y), (s, t) \in \mathbb{R}^2$ we have*

$$|\Delta_{s,t} f(x, y)| \leqslant \frac{\varepsilon}{3} + A(\varepsilon) \frac{\sin^2(x - s)}{2} + B(\varepsilon) \frac{\sin^2(y - t)}{2}.$$

Proof. Let ε be a given positive real number. Because f is uniformly B-continuous on $\mathbb{R}^2$ by Lemma 2.5 there exists a $\delta(\varepsilon) \in]0, \pi]$ such that for every $(x_1, y_1),(x_2, y_2) \in \mathbb{R}^2$ with $|x_1 - x_2| \leqslant \delta(\varepsilon)$ and $|y_1 - y_2| \leqslant \delta(\varepsilon)$ we have

$$|\Delta_{x_2,y_2} f(x_1, y_1)| \leqslant \frac{\varepsilon}{3}. \tag{3.1}$$

Given $(x, y),(s, t) \in \mathbb{R}^2$ we choose $k, l \in \mathbb{Z}$ such that for $(x', y') = (x + 2k\pi, y + 2l\pi)$ there holds

$$|x' - s| \leqslant \pi, \quad |y' - t| \leqslant \pi.$$

We distinguish the following four situations

(*i*) $|x' - s| \leqslant \delta(\varepsilon), \ |y' - t| \leqslant \delta(\varepsilon)$;

(*ii*) $|x' - s| > \delta(\varepsilon), \ |y' - t| \leqslant \delta(\varepsilon)$;

(*iii*) $|x' - s| \leqslant \delta(\varepsilon), \ |y' - t| > \delta(\varepsilon)$;

(*iv*) $|x' - s| > \delta(\varepsilon), \ |y' - t| > \delta(\varepsilon)$.

In case (i), using (3.1), we have

$$|\Delta_{s,t} f(x', y')| \leqslant \frac{\varepsilon}{3}. \tag{3.2}$$

In case (ii), there holds

$$\sin \frac{\delta(\varepsilon)}{2} \leqslant \sin \frac{x' - s}{2}.$$

So we have

$$\frac{\sin^2 \dfrac{x' - s}{2}}{\sin^2 \dfrac{\delta(\varepsilon)}{2}} \geqslant 1. \tag{3.3}$$

Using Lemma 2.3 and (3.3) we find that

$$|\Delta_{s,t} f(x', y')| \leqslant \frac{M}{\sin^2 \dfrac{\delta(\varepsilon)}{2}} \cdot \sin^2 \frac{x' - s}{2}. \tag{3.4}$$

In the case (iii) and (iv) we obtain in a similar manner

$$|\Delta_{s,t} f(x', y')| \leqslant \frac{M}{\sin^2 \dfrac{\delta(\varepsilon)}{2}} \cdot \sin^2 \frac{y' - t}{2}, \tag{3.5}$$

and

$$|\Delta_{s,t} f(x', y')| \leqslant \frac{M}{\sin^4 \dfrac{\delta(\varepsilon)}{2}} \cdot \sin^2 \frac{x' - s}{2} \sin^2 \frac{y' - t}{2} \leqslant \frac{M}{\sin^4 \dfrac{\delta(\varepsilon)}{2}} \cdot \sin^2 \frac{x' - s}{2}.$$

3.6)

(Consequently, employing (3.2), (3.4), (3.5), and (3.6) we have the following inequality

$$|\Delta_{s,t} f(x', y')| \leqslant \frac{\varepsilon}{3} + \left(\frac{M}{\sin^2 \dfrac{\delta(\varepsilon)}{2}} + \frac{M}{\sin^4 \dfrac{\delta(\varepsilon)}{2}} \right) \cdot \sin^2 \frac{x' - s}{2} + \frac{M}{\sin^2 \dfrac{\delta(\varepsilon)}{2}} \cdot \sin^2 \frac{y' - t}{2}. \tag{3.7}$$

In consideration of Lemma 2.2 and of the periodicity properties of the sine function we may substitute x' by x and y' by y in (3.7) which completes the proof. ∎

Now we are ready to show

THEOREM 3.2. *Let $(L_{m,n})$, $m, n \in \mathbb{N}$, be a sequence of positive linear operators transforming bounded $B_{2\pi}$-functions which are 2π-periodic with respect to both variables into functions of $\mathbb{R}^{\mathbb{R}^2}$ and satisfying*

(*i*) $L_{m,n}(e; x, y) = 1$, *where* $e(s, t) = 1$.

For $f \in B_{2\pi}$ *and* $(x, y) \in \mathbb{R}^2$ *let*

$$U_{m,n} f(x, y) := L_{m,n}(f(\cdot, y) + f(x, \star) - f(\cdot, \star); x, y).$$

If the conditions

(*ii*) $L_{m,n}(\varphi_1; x, y) = \sin x + u_{m,n}(x, y)$,

(*iii*) $L_{m,n}(\varphi_2; x, y) = \sin y + v_{m,n}(x, y)$,

(*iv*) $L_{m,n}(\psi_1; x, y) = \cos x + t_{m,n}(x, y)$,

(*v*) $L_{m,n}(\psi_2; x, y) = \cos y + w_{m,n}(x, y)$,

where $\varphi_1(s, t) := \sin s$, $\varphi_2(s, t) := \sin t$, $\psi_1(s, t) := \cos s$, $\psi_2(s, t) := \cos t$, *and* $u_{m,n}$, $v_{m,n}$, $t_{m,n}$ *and* $w_{m,n}$ *converge to zero uniformly on* $\mathbb{R}^2$ *as* m, n *approach infinity, are satisfied, then for every* $f \in B_{2\pi}$ *the sequence* $(U_{m,n} f)$ *converges uniformly to* f *on* $\mathbb{R}^2$.

Proof. Let $f \in B_{2\pi}$ and $(x, y) \in \mathbb{R}^2$. Because of condition (i) we may write

$$U_{m,n} f(x, y) = f(x, y) - L_{m,n}(\Delta_{x,y} f; x, y). \tag{3.8}$$

By the results of section 2 (cf. especially Corollary 2.4 and the proof thereof) we see that $U_{m,n}$ is a well-defined linear operator on $B_{2\pi}$.

Furthermore, taking into account the positivity of $L_{m,n}$, Lemma 3.1 implies

$$|f(x,y) - U_{m,n}f(x,y)| = \max\{L_{m,n}(\Delta_{x,y}f;x,y), L_{m,n}(-\Delta_{x,y}f;x,y)\}$$
$$\leqslant L_{m,n}\left(\frac{\varepsilon}{3} + A(\varepsilon)\sin^2\frac{x-\cdot}{2} + B(\varepsilon)\sin^2\frac{y-\star}{2};x,y\right)$$

for each positive number ε.

Carrying through some operations and applying the relations (i) to (v) from the statement of the theorem we arrive at the estimate

$$|f(x,y) - U_{m,n}f(x,y)|$$
$$\leqslant \frac{\varepsilon}{3} + \frac{1}{2}c(\varepsilon)\cdot\{2 - \cos x\cdot L_{m,n}(\psi_1;x,y) - \sin x\cdot L_{m,n}(\varphi_1;x,y) -$$
$$- \cos y\cdot L_{m,n}(\psi_2;x,y) - \sin y\cdot L_{m,n}(\varphi_2;x,y)\}$$
$$= \frac{\varepsilon}{3} - \frac{1}{2}c(\varepsilon)\cdot\{\cos x\cdot t_{m,n}(x,y) + \sin x\cdot u_{m,n}(x,y) +$$
$$+ \cos y\cdot w_{m,n}(x,y) + \sin y\cdot v_{m,n}(x,y)\},$$

where $c(\varepsilon) = \max\{A(\varepsilon), B(\varepsilon)\}$.

Letting m and n tend to infinity yields the desired result. ∎

Remarks 3.3

(a) A Korovkin-type theorem for the convergence of the operators $(L_{m,n})$ themselves in the smaller space $C_{2\pi,2\pi}$ can be found in [13].

(b) If we choose $L_{m,n}$ in the above theorem as a product of the parametric extensions of two univariate positive linear operators with suitable domains, then $U_{m,n}$ is the Boolean sum of these univariate operators. Thus it is justified to call our operators a generalization of Boolean sum operators.

(c) If equality (i) of the theorem does not hold, then equation (3.8) is not true. If one replaces (i) by (i′) $L_{m,n}(e;x,y) = 1 + \alpha_{m,n}(x,y)$ then the above method of proof allows only the conclusion that we have pointwise convergence to $f(x,y)$ for all $(x,y)\in\mathbb{R}^2$, if $\alpha_{m,n}(x,y)$ converges uniformly to zero as m, n tend to infinity (cf. the remark following the main result in [2]).

(d) Note that with the same argument as in the proof of Theorem 3.2 also for the Korovkin-type theorems in [1] and [2] it would be sufficient to require that $L_{m,n}$ is defined for *bounded* B-continuous functions.

(e) The class of operators defined for all B-continuous functions even with the additional assumption of boundedness, is somewhat restricted. For example Theorem 3.2 may not directly be applied to such important operators as integral convolution operators since a B-continuous function need not be measurable. We remark however, that it is no problem to formulate theorems similar to Theorem 3.2 for operators with some other domains, e.g. spaces of (bounded) measurable $B_{2\pi}$-functions or spaces of $C_{2\pi,2\pi}$-functions. The same remark again applies to the Korovkin theorems in [1] and [2].

4. Approximation with trigonometric pseudopolynomials. While it is clear that functions in $C_{2\pi,2\pi}$ can be approximated arbitrarily well by functions of the form (1.1) in the uniform norm, this fact is not so obvious for the larger space $B_{2\pi}$.

However, it can be obtained as a consequence of Theorem 3.2. The idea is to take the Boolean sum of parametric extensions of univariate polynomial approximation operators as pseudopolynomial approximation operators. This kind of procedure was also used in [2] to prove an approximation theorem for B-continuous functions by algebraic pseudopolynomials. While the underlying approximation problem was not new in the algebraic case (cf. the references in [2]), to our knowledge we are the first to consider the corresponding trigonometric problem.

In the trigonometric case the best-known examples of univariate polynomial approximation operators are of integral type so that the Boolean sum would only be defined on a subspace of $B_{2\pi}$. But we can also use suitable discretely defined operators which are defined for all real-valued functions.

Before considering any special situations we look at two arbitrary discretely defined and constant-reproducing positive linear operators L_m and $\bar{L}_n$, i.e. operators given by

$$L_m(f;x) = \sum_{i=0}^{m} f(x_i)\cdot p_{m,i}(x),$$
$$\bar{L}_n(g;y) = \sum_{j=0}^{n} g(y_j)\cdot q_{n,j}(y), \tag{4.1}$$

where $\sum_{i=0}^{m} p_{m,i}(x) = \sum_{j=0}^{n} q_{n,j}(y) = 1$ for all m, n and all $x, y\in\mathbb{R}$, and

$$p_{m,i}(x) \geqslant 0 \text{ for } 0\leqslant i\leqslant m \text{ and all } x\in\mathbb{R},$$

$$q_{n,j}(y) \geqslant 0 \text{ for } 0\leqslant j\leqslant n \text{ and all } y\in\mathbb{R}.$$

Now the tensor product $L_{m,n} = L_m^x\circ\bar{L}_n^y$, where $L_m^x = L_m$ acts on the bivariate function $f(x,y)$ as if y is a fixed parameter, and $\bar{L}_n^y = \bar{L}_n$ acts on $f(x,y)$ as if x is fixed, is a positive linear and constant-reproducing operator defined for any function $f:\mathbb{R}^2\to\mathbb{R}$. The operator $U_{m,n}$ defined in Theorem 3.2 in this case equals the Boolean sum $L_{m,n} = L_m^x \oplus \bar{L}_n^y$ (cf. Remark 3.3(b)), and the conditions (ii)–(v) in Theorem 3.2 may be replaced by the conditions

(ii′) $L_m(\varphi;x) = \sin x + u_m(x)$,
(iii′) $\bar{L}_n(\varphi;y) = \sin y + v_n(y)$,
(iv′) $L_m(\psi;x) = \cos x + t_m(x)$,
(v′) $\bar{L}_n(\psi;y) = \cos y + w_n(y)$,

where $\varphi(z) := \sin z$, $\psi(z) := \cos z$, and u_m, t_m, v_n, w_n converge to zero uniformly on R as m and n approach infinity.

The preceeding discussion yields

LEMMA 4.1. *If the sequences $(L_m)_{m\in\mathbb{N}}$, $(\bar{L}_n)_{n\in\mathbb{N}}$ of positive linear operators are given as in (4.1), and if $L_m\varphi \to \varphi$, $\bar{L}_n\varphi \to \varphi$, $L_m\psi \to \psi$, $\bar{L}_n\psi \to \psi$ uniformly on $\mathbb{R}$, then the operators $U_{m,n}$ constructed on the basis of $L_{m,n} = L_m^x \circ \bar{L}_n^y$ have the property that $(U_{m,n}f)$ converges uniformly to f for each $f \in B_{2\pi}$, as m and n tend to infinity.* ▮

To obtain the result announced at the beginning of this section we can now use for instance a sequence of operators (K_n) of the form

$$K_n(f;x) = \frac{2}{N_n+2}\sum_{k=1}^{N_n+2} f(t_{k,n})\cdot \Phi_n(t_{k,n}-x)$$

where $t_{k,n} = \dfrac{2k\pi}{N_n+2}$, $k = 1, 2, \ldots, N_n + 2$, and Φ_n is a nonnegative cosine polynomial of the form

$$\Phi_n(x) = \frac{1}{2} + \sum_{\nu=1}^{N_n} \rho_{\nu,n}\cdot \cos \nu x, \lim_{n\to\infty} \rho_{1,n} = 1.$$

Examples of such operators can be found e.g. in [7]. As other rather new references for discrete linear polynomial approximation operators we cite the papers [9] and [10].

Theorem B in [7] shows that K_n reproduces constants, and that $K_n f$ converges uniformly to f as n tends to infinity for all f in the space of (univariate) continuous, 2π-periodic functions, i.e. especially for the functions $\sin x$, $\cos x$.

Now consider the Boolean sum operator $W_{m,n}$ defined by

$$W_{m,n}f(x,y) = (K_m^x \oplus K_n^y)f(x,y)$$

$$= \frac{4}{(N_m+2)(N_n+2)}\sum_{k_1=1}^{N_m+2}\sum_{k_2=1}^{N_n+2}[f(x,t_{k_2,n}) + f(t_{k_1,m},y) - f(t_{k_1,m},t_{k_2,n})] \cdot \Phi_m(t_{k_1,m}-x)\cdot\Phi_n(t_{k_2,n}-y).$$

The definition of K_n and the results cited above show that $W_{m,n}f$ is a trigonometric pseudopolynomial and that $W_{m,n}$ satisfies the assumptions of Theorem 3.2. Thus we have

COROLLARY 4.2. *If $f \in B_{2\pi}$ then $(W_{m,n}f)$ converges uniformly to f on $\mathbb{R}^2$, i.e. for each $f \in B_{2\pi}$ there exists a sequence of uniformly approximating rigonometric pseudopolynomials.* ▮

We are now interested in the question whether it is possible to get some additional information on the coefficient functions of the approximating pseudopolynomials in Corollary 4.2. It is clear that it is possible to choose continuous coefficient functions if the approximated function f is continuous. Moreover, if f is bounded, also $W_{m,n}f$ is bounded, and the boundedness of a pseudopolynomial is equivalent to the boundedness of its coefficient functions. For the trigonometric case we can show this fact similarly as was done for the algebraic case in [14].

Bounded pseudopolynomials are also called *Marchaud pseudopolynomials* after. A. Marchaud who first examined them in [11], [12]. Since the limit of a uniformly converging sequence of bounded functions is still bounded it is clear that a general $f \in B_{2\pi}$ cannot be approximated by (trigonometric) Marchaud pseudopolynomials.

However, in view of Corollary 2.4 it actually suffices to consider the approximation problem for bounded $B_{2\pi}$-functions: We first approximate the bounded part of f by a Marchaud pseudopolynomial and then add to it the B-constant part of f which is obviously a pseudopolynomial, too. If we call the sum of a Marchaud pseudopolynomial and a B-constant a *B-Marchaud pseudopolynomial* we have

COROLLARY 4.3. *Each $f \in B_{2\pi}$ may be uniformly approximated by a sequence of B-Marchaud pseudopolynomials.* ▮

Note that as an extension of the results in [2, Section 4] a similar assertion can also be proved in the algebraic case.

5. Concluding Remark. A natural continuation of the considerations of this paper consists of quantitative versions of Theorem 3.2, similar to the quantitative version of its algebraic analogon in [1]. Statements of this type will be contained in the doctoral dissertation of the third author.

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