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ON THE APPROXIMATE SOLUTION OF OPERATOR EQUATIONS IN HILBERT SPACE BY A STEFFENSEN-TYPE METHOD

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0. The classical secant method can be extended for approximate solving a non-linear operator equation defined in Banach spaces [4]. This method assumes the existence and the continuity of the inverse of the divided difference [3]. But the examination of the existence of this inverse operator and the evoluation of its norm are of great difficulties. In the papers [1], [2] the author presented methods for approximate solution of non-linear operator equations in Hilbert spaces, which eliminates the above-mentioned difficulties.

In this paper we give a Steffensen-type method for approximate solution of non-linear operator equations in Hilbert real spaces, which neither supposes the existence of the inverse of the divided difference of the operator.

1. Let H be a Hilbert real space and let $\langle v_1, v_2\rangle$ be the scalar product of the vectors $v_1, v_2 \in H$. Consider the non-linear equation

$$P(x) = 0 \tag{1}$$

and the equivalent equation

$$\| P(x) \|^2 = \langle P(x), P(x)\rangle = 0 \tag{2}$$

where P is a continuous operator defined on H with values in H, $P : H \to H$. Let $[x', x''; P]$ be a simmetrical divided difference of the operator P for the points $x', x'' \in H$, $x' \neq x''$. (It is known that every operator P has such a divided difference [3].) In this case the linear mapping given by $x \mapsto \langle [x', x''; P](x), P(x') + P(x'')\rangle$ is a simmetrical divided difference of the functional $\| P \|^2 : H \to R$ [1]. Let's define a non-linear operator $F : H \to H$ using the above mentioned functional $\| P \|^2$

$$F(x) = x - \| P(x) \| \frac{x}{\| x \|}, \quad x \neq 0 \tag{3}$$

and (x_n), $x_n \in H$ a vector sequence. Consider the sequence (u_n), $u_n = F(x_n) \in H$ and the sequence (z_n), $z_n \in H$ such that

$$\| z_n \| = 1 \quad \text{and} \tag{4}$$

$$| [x_n, u_n ; \| P \|^2](z_n) | = \| [x_n, u_n ; \| P \|^2] \|.$$

Let us note that such a choice of z_n is possible [1].

THEOREM. *Suppose that the following conditions hold:*
1° *there exists the point* $x_0 \in H$ *and the constant* $B_0 > 1$ *such that*

$$B_0 \| \langle [x_0, u_0 ; P], P(x_0) + P(u_0) \rangle \| \geq 1 ;$$

2° *there exists the constant* $\eta_0 > 0$ *such that the following inegality is satisfied:*

$$\| P(x_0) \|^2 \leq \eta_0 ;$$

3° *the divided difference of the operator* F *and the divided difference of second order of the functional* $\| P \|^2$ *satisfy the following relations:*

$$\| [x', x'' ; F] \| \leq M, \quad \| [x', x'', x''' ; \| P \|^2] \| \leq K$$

if $x', x'', x''' \in S[x_0, r]$, *where* $S[x_0, r]$ *is the ball with centre* x_0 *and radius* $r = 3B_0\eta_0$;
4° *the constants* B_0, η_0, K, M *satisfy the inequality*

$$h_0 := 2B^2K(M+1)\,\eta_0 < \frac{1}{3}.$$

Then the equation (1) *has at least one solution* $x^* \in S[x_0, r]$ *which is the limit of the approximations given by the equality*

$$x_{n+1} = x_n - \frac{\| P(x_n) \|^2}{\langle [x_n, u_n ; P](z_n), P(x_n) + P(u_n) \rangle} \cdot z_n \tag{5}$$

$n = 0, 1, 2, \ldots,$ *and for the error estimate we have the inequality*

$$\| x^* - x_n \| \leq \left(\frac{2}{3} \right)^n \left(\frac{3}{4} \right)^{2^n - 1} \cdot s$$

where

$$s := B_0 \eta_0 \sum_{k=0}^{\infty} \left(\frac{2}{3} \right)^k \left(\frac{3}{4} \right)^{2(2^{k-1}-1)}$$

Proof. From the formulas (3), (5) and the conditions 1°—3° of Theorem it results:

$$\| x_0 - u_0 \| = \| x_0 - x_0 + \| P(x_0) \|^2 \frac{x_0}{\| x_0 \|} = \| P(x_0) \|^2 \leq \eta_0 < r,$$

$$\| x_0 - x_1 \| = \frac{\| P(x_0) \|}{|\langle [x_0, u_0 ; P](z_0), P(x_0) + P(u_0) \rangle|} \cdot \| z_0 \| \leq B_0\eta_0 < r,$$

$$\| u_0 - u_1 \| = \| F(x_0) - F(x_1) \| = \| [x_1, x_0 ; F](x_1 - x_0) \| \leq MB_0\eta_0, \tag{6}$$

$$\| x_0 - u_1 \| \leq \| x_0 - u_0 \| + \| u_0 - u_1 \| \leq \eta_0 + MB_0\eta_0 \leq B_0\eta_0(1 + M) \leq r,$$

$$\| x_1 - u_0 \| \leq \| x_1 - x_0 \| + \| x_0 - u_0 \| \leq B_0\eta_0 + \eta_0 \leq 2B_0\eta_0.$$

The conditions (6) show that the points u_0, x_1, u_1 are in the ball $S[x_0, r]$.

Using the definition of the divided difference of second order, the condition 1° of Theorem and the inequality (6), we obtain

$$\frac{\| [x_1, u_1 ; \| P \|^2] - [x_0, u_0 ; \| P \|^2] \|}{\| [x_0, u_0 ; \| P \|^2] \|} \leq$$

$$\leq B_0 \| [x_1, u_1 ; \| P \|^2] - [x_1, u_0 ; \| P \|^2] + [x_1, u_0 ; \| P \|^2] -$$

$$- [x_0, u_0 ; \| P \|^2] \| = B_0 \| [x_1, u_1, u_0 ; \| P \|^2](u_1 - u_0) +$$

$$+ [x_1, x_0, u_0 ; \| P \|^2](x_1 - x_0) \| \leq B_0 K (\| x_1 - x_0 \| + \| u_1 - u_0 \|) \leq$$

$$\leq KB_0^2\eta_0(1 + M) < h_0.$$

Fom the evident inequality

$$\| [x_1, u_1 ; \| P \|^2] \| \geq \| [x_0, u_0 ; \| P \|^2] \| \left(1 - \frac{\| [x_1, u_1 ; \| P \|^2] - [x_0, u_0 ; \| P \|^2] \|}{\| [x_0, u_0 ; \| P \|^2] \|} \right)$$

using the above inequality, it results:

$$\frac{1}{\| [x_1, u_1 ; \| P \|^2] \|} \leq \frac{B_0}{1 - h_0} := B_1 \leq \frac{3}{2} B_0$$

whence we obtain

$$B_1 \| \langle [x_1, u_1 ; P], P(x_1) + P(u_1) \rangle \| \geq 1, \quad B_1 \geq 1. \tag{7}$$

Using equality (4) from the formula (5), for $n = 0$ it results immediatly

$$[x_0, u_0 ; \| P \|^2](x_0 - x_1) = \| P(x_0) \|^2,$$

whence we get

$$\| P(x_1) \|^2 = \| P(x_1) \|^2 - \| P(x_0) \|^2 - [x_0, u_0 ; \| P \|^2](x_1 - x_0) =$$

$$= [x_1, x_0 ; \| P \|^2](x_1 - x_0) - [x_0, u_0 \| P \|^2](x_1 - x_0) =$$

$$= [x_1, x_0, u_0 ; \| P \|^2](x_1 - u_0)(x_1 - x_0).$$

Therefore, according to (6) and condition 3° of the Theorem, we obtain

$$\| P(x_1) \|^2 \leq K \| x_1 - u_0 \| \cdot \| x_1 - x_0 \| \leq 2KB_0^2\eta_0^2 < h_0\eta_0 := \eta_1 \tag{8}$$

Considering the formulas (7), (8) we have

$$h_1 := 2B_1^2K(M+1)\,\eta_1 \leq \frac{h_0^2}{(1 - h_0)^2} < \frac{1}{3}. \tag{9}$$

The inequalities (7), (8), (9) show that the conditions 1°, 2° and 4° of the Theorem are satisfied for the points x_1, u_1.

By mathematical induction we can prove the existence of the constants b_n, η_n such that the following inequalities hold

$$\frac{1}{\| [x_n, u_n ; \| P \|^2] \|} \leqslant B_n = \frac{B_{n-1}}{1 - h_{n-1}} \leqslant \left(\frac{3}{2}\right)^n B_0,$$

$$\| P(x_n) \|^2 \leqslant \eta_n = h_{n-1}\, \eta_{n-1} \leqslant \left(\frac{2}{3}\right)^{2n} \left(\frac{3}{4}\right)^{2^n - 1} \eta_0, \tag{10}$$

$$h_n \leqslant \frac{h_{n-1}^2}{(1 - h_{n-1})^2} \leqslant \left(\frac{3}{2}\right)^{2(2^n - 1)} h_0^{2^n}.$$

On the basis of the formulas (10) we have

$$\begin{aligned} \| x_{n+p} - x_n \| &\leqslant \| x_{n+1} - x_n \| + \ldots + \| x_{n+p-1} - x_{n+p} \| \leqslant \\ &\leqslant B_n \eta_n + \ldots + B_{n+p-1} \eta_{n+p-1} \leqslant \\ &\leqslant B_0 \eta_0 \sum_{k=0}^{p-1} \left(\frac{2}{3}\right)^{n+k} \left(\frac{3}{4}\right)^{2^{n+k} - 1} = \\ &= B_0 \eta_0 \left(\frac{2}{3}\right)^n \cdot \left(\frac{3}{4}\right)^{2^n - 1} \sum_{k=0}^{p-1} \left(\frac{2}{3}\right)^k \left(\frac{3}{4}\right)^{2(2^k - 1)} < \\ &< \left(\frac{2}{3}\right)^n \left(\frac{3}{4}\right)^{2^n - 1} \cdot s \end{aligned} \tag{11}$$

where s is the sum of the convergent series

$$B_0 \eta_0 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \left(\frac{3}{4}\right)^{2(2^k - 1)}.$$

Because the space H is a Banach space, it results the existence of the limit of the sequence (x_n), and $x^* \in H$. Since $\lim\limits_{n \to \infty} \eta_n = 0$ by the continuity of the operator P we obtain $P(x^*) = 0$. From (11) if $p \to \infty$ it results the inequality which we have for the error estimate.

Using inequalities (10), we get that the points x_n, u_n and x^* are in the ball $S[x_0, r]$. Indeed, we have

$$\begin{aligned} \| x_n - x_0 \| &\leqslant \| x_0 - x_1 \| + \| x_1 - x_2 \| + \ldots + \| x_{n-1} - x_n \| \leqslant \\ &\leqslant B_0 \eta_0 + B_1 \eta_1 + \ldots + B_{n-1} \eta_{n-1} \leqslant \\ &\leqslant B_0 \eta_0 \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k \left(\frac{3}{4}\right)^{2^k - 1} \leqslant \\ &\leqslant B_0 \eta_0 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \left(\frac{3}{4}\right)^k = 2 B_0 \eta_0 \end{aligned}$$

$$\begin{aligned} \| u_n - x_0 \| &< \| x_n - x_0 \| + \| x_n - u_n \| \leqslant 2 B_0 \eta_0 + \| P(x_n) \|^2 < \\ &< 2 B_0 \eta_0 + \eta_0 < 3 B_0 \eta_0. \end{aligned}$$

Choosing r as in condition 3° of the Theorem it results $x^* \in S[x_0, r]$.

Remark. The condition 3° of the Theorem can be replaced by the following: $\| \langle [x', x''; P], P(x') + P(x'') \rangle - \langle [x'', x'''; P], P(x'') + P(x''') \rangle \| \leqslant K \| x''' - x' \|$,
namely the divided difference of the functional $\| P \|^2$ has the Liepschitz-property in $S[x_0, r]$.

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