MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 17, N° 1, 1988, pp. 29—32

AN ALGORITHM FOR DETERMINING THE BASES OF A REGULAR MATROID

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The regular matroids mark an interesting half-way stage between the matroids corresponding to graphs on the one hand, and the binary matroids, corresponding to chain-groups over GF(2), on the other (see [1]).
In this note we give a matrix algorithm for finding all bases of a regular matroid.

The terminology used in this paper is that of [1] and [2]. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a finite set and M a *regular* matroid on E whose rank is r. We shall denote by $\widetilde{W}$ the *weakly representative matrix* of M. If S is any subset of E we define $\widetilde{W}(S)$ as the submatrix of $\widetilde{W}$ consisting of those columns that correspond to members of S. If $|S| = r$ it may happen that $\widetilde{W}(S)$ is a *diagonal matrix*, that is, the diagonal elements are all nonzero and the nondiagonal elements are all zero. We then say that $\widetilde{W}$ is in *diagonal form* with respect to S.

Let $\widetilde{W}$ be a weakly representative matrix of M diagonal with respect to a subset S of E. It may happen that $\widetilde{W}(S)$ is the unit matrix. In this case $\widetilde{W}$ is a true representative matrix of M (see [1]) and is called *standard* representative matrix of M associated with S.

THEOREM 1 (W. T. Tutte, [2]). *If W is a standard representative matrix of M, then W is completely unimodular.*

THEOREM 2 (W. T. Tutte, [2]). *Let W be a standard representative matrix of M and B a subset of E. The determinant of $W(B)$ has one value 1 or -1 if B is a basis of M and 0 otherwise.*

By standard results of linear algebra the property of being a representative matrix of M is invariant (see [1]) under the following "elementary operations":

O_1. Permuting two rows (or columns).

O_2. Adding to one row (or column) a multiple of another by 1 or -1.

O_3. Multiplying a row (or column) by -1.

Let $\widetilde{W}$ be any weakly representative matrix of M and S a subset of E of r elements. Then we can find a standard representative matrix of M

associated with S if and only if $\widetilde{W}(S)$ is nonsingular (see [1]), i.e., by theorems 1 and 2, if and only if S is a basis of M.

Let B_0 be a fixed basis of M and $k \leqslant r$ a nonnegative integer. We shall denote by $B(M, B_0, k)$ the set of bases of M for which $| B_0 - B | = k$ for every $B \in B(M, B_0, k)$. Without loss of generality let B_0 be $B_0 = \{e_1, e_2, \ldots, e_r\}$ and W the standard representative matrix of M associated to B_0 (W can be obtained using $O_1 - O_3$). Thus W is of the form

$$W = [I_r \mid W_{r,m}], \tag{1}$$

where I_r is the unit matrix of order r and $m = n - r$.

Let $i \in \{1, 2, \ldots r\}$ and $j \in \{r+1, r+2, \ldots, n\}$. The expansion of $\det W[(B_0 - \{e_i\}) \cup \{e_j\}]$, using the rows in $W[(B_0 - \{e_i\}) \cup \{e_j\}]$, shows that $\det W[(B_0 - \{e_i\}) \cup \{e_j\}] = W_{ij}$ (see (1)). Therefore, by theorems 1 and 2, we have:

$$(B_0 - \{e_i\}) \cup \{e_j\} \in B(M, B_0, 1) \text{ if and only if } W_{ij} \neq 0. \tag{2}$$

Let $1 < k \leqslant r$ be fixed, $g_i = e_{r+i}$, $i = 1, 2, \ldots, m$ and W partitioned as follows:

$$W = \left[\begin{array}{c:c:c:c} I_k & O_{k,r-k} & W^a_{k,k} & W^b_{k,m-k} \\ \hdashline O_{r-k,k} & I_{r-k,r-k} & W^c_{r-k,k} & W^d_{r-k,m-k} \end{array}\right],$$

where O_{pq} denotes the null matrix with p rows and q columns and W_{pq} denotes a submatrix of W with p rows and q columns.

Interchanging the columns corresponding to $e_1, e_2, \ldots, e_k$ with the columns corresponding to $g_1, g_2, \ldots, g_k$ respectively, we obtain the matrix:

$$W' = \left[\begin{array}{c:c:c:c} W^a & O_{k,r-k} & I_k & W^b \\ \hdashline W^c & I_{r-k,r-k} & O_{r-k,k} & W^d \end{array}\right].$$

Obviously, the matrix $\left[\begin{array}{c:c} W^a & O_{k,r-k} \\ \hdashline W^c & I_{r-k,r-k} \end{array}\right]$ is nonsingular if and only if W^a is nonsingular. Hence, by theorems 1 and 2, the set $\{g_1, g_2, \ldots, g_k, e_{k+1}, e_{k+2}, \ldots, e_r\}$ is a basis of M if and only if W^a is nonsingular. Therefore, for obtain all bases of M by replacing $\{e_1, e_2, \ldots, e_k\}$ in B_0 we must find all nonsingular square (of order k) submatrices of the matrix

$$\widetilde{W}(e_1, e_2, \ldots, e_k) = [W^a \mid W^b].$$

It is easy to see that $\widetilde{W}(e_1, e_2, \ldots, e_k)$ is a weakly representative matrix of the minor $M(e_1, e_2, \ldots, e_k) = [M \cdot (E - \{e_{k+1}, e_{k+2}, \ldots, e_r\})] \times (E - \{e_1, e_2, \ldots, e_k\})$ of M. By [1], $M(e_1, e_2, \ldots, e_k)$ is also a regular matroid and the columns of every nonsingular submatrix of order k of the matrix $\widetilde{W}(e_1, e_2, \ldots, e_k)$ corresponds to a basis of $M(e_1, e_2, \ldots, e_k)$. Obviously, the union of every basis of $M(e_1, e_2, \ldots, e_k)$ with the set $\{e_{k+1}, e_{k+2}, \ldots, e_r\}$ is a basis of M. Thus, for obtain all bases of M by replacing $\{e_1, e_2, \ldots, e_k\}$ in B_0 it is sufficiently to find all bases of $M(e_1, e_2, \ldots, e_k)$.

It is also easy to see that using $O_1 - O_3$ the matrix $\widetilde{W}(e_1, e_2, \ldots, e_k)$ can be transformed in a standard representative matrix of M denoted $W(e_1, e_2, \ldots, e_k)$. We shall denote by $B(B_0, e_1, e_2, \ldots, e_k)$ the set of all bases of M obtained by replacing $\{e_1, e_2, \ldots, e_k\}$ in B_0. Summarizing we have the following

Algorithm

Step 1. Starting with a weakly representative matrix $\widetilde{W}$ of M and with a basis B_0 we obtain (using $O_1 - O_3$) a standard representative matrix W of M associated to B_0.

Step 2. We obtain $B(M, B_0, 1)$ according to (2).

Step 3. For every combination of $k > 1$ elements $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ with $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant r$ we find $B(B_0, e_{i_1}, e_{i_2}, \ldots, e_{i_k})$. For determining all members of $B(B_0, e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ we shall consider all the nonsingular square (of order k) submatrices of the matrix $\widetilde{W}(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ obtained with the columns of $W_{r,m}$ and the rows of W containing 1 in the column e_{i_s}, $s = 1, 2, \ldots, k$. For the simplicity, the matrix $\widetilde{W}(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ can be considered as a weakly representative matrix of $M(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ for which we reconsider the problem to finding all bases by repeating (if it is necessary) the steps 1, 2 and 3 starting with the step 1. To each basis obtained for $M(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ corresponds a basis of M by adding $B_0 - \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$.

REMARK. If, for a $k > 1$, we denote by $I(k)$ the set of all combinations as the form $(1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant r)$ we then have:

$$B(M, B_0, k) = \bigcup_{(i_1, i_2, \ldots, i_k) \in I(k)} B(B_0, e_{i_1}, e_{i_2}, \ldots, e_{i_k}).$$

It is also easy to see that the above algorithm does not produce duplications and generates all bases of M.

Worked Example. Let $E = \{a, b, c, d, e, f\}$ and M a matroid on E whose family of circuits is $C(M) = \{\{b, c\}, \{a, b, d\}, \{a, c, d\}, \{d, e, f\}, \{a, b, e, f\}, \{a, c, e, f\}\}$. It is easy to check that the following matrix

$$\widetilde{W} = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

is a weakly representative matrix of M. Let $B_0 = \{a, b, e\}$ be a fixed basis of M.

Step 1. Using $O_1 - O_3$ we obtain the following standard representative matrix of M:

$$W = \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad (\text{columns } e_1, e_2, e_3, e_4, e_5, e_6),$$

where $e_1 = a$, $e_2 = b$, $e_3 = e$, $e_4 = f$, $e_5 = c$, $e_6 = d$ and $B_0 = \{e_1, e_2, e_3\}$.

Step 2. According to (2) we have $B(M, B_0, 1) = \{\{e_4, e_2, e_3\}, \{e_6, e_2, e_3\}, \{e_1, e_4, e_3\}, \{e_1, e_5, e_3\}, \{e_1, e_6, e_3\}, \{e_1, e_2, e_4\}\}$.

Step 3. For finding $B(M, B_0, 2)$ we repeat the steps 1 and 2 as follows:

$$\widetilde{W}(e_1, e_2) = \begin{array}{c} \\ e_1 \\ e_2 \end{array}\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \end{array}, \quad W(e_1, e_2) = \begin{array}{c} \\ e_4 \\ e_5 \end{array}\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \end{array},$$

$B_0^1 = \{e_4, e_5\}$ and $B[M(e_1, e_2), B_0^1, 1] = \{\{e_6, e_5\}\}$;

$$\widetilde{W}(e_1, e_3) = \begin{array}{c} \\ e_1 \\ e_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad W(e_1, e_3) = \begin{array}{c} \\ e_6 \\ e_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_6 & e_4 & e_5 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array},$$

$B_0^2 = \{e_6, e_4\}$ and $B[M(e_1, e_3), B_0^2, 1] = \emptyset$;

$$\widetilde{W}(e_2, e_3) = \begin{array}{c} \\ e_2 \\ e_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad W(e_2, e_3) = \begin{array}{c} \\ e_6 \\ e_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_6 & e_4 & e_5 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right] \end{array},$$

$B_0^3 = \{e_6, e_4\}$ and $B[M(e_2, e_3), B_0^3, 1] = \{\{e_5, e_4\}\}$.

Thus we have $B(M, B_0, 2) = \{\{e_4, e_5, e_3\}, \{e_6, e_5, e_3\}, \{e_6, e_4, e_2\}, \{e_6, e_4, e_1\}, \{e_5, e_4, e_1\}\}$. For finding $B(M, B_0, 3)$ we must consider:

$$\widetilde{W} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad W = \begin{array}{c} \\ e_6 \\ e_5 \\ e_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_6 & e_5 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array},$$

$$B_0^4 = \{e_6, e_5, e_4\}.$$

Now the process is finished and all bases of M are generated without duplications (the matroid of our example contains 13 bases).

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Received 1.XI.1987

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