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MAXIMAL PSEUDOMONOTONICITY OF GENERALIZED SUBDIFFERENTIALS OF EXPLICITLY QUASICONVEX FUNCTIONS

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It is well known that the subdifferential of any proper function f from a Banach space X to $\overline{\mathbb{R}}$ is a cyclically monotone mapping and if f is lower-semicontinuous, proper and convex, then it is subdifferentiable at each interior point of its effective domain. Moreover, on these assumptions, the subdifferential of f is a maximal cyclically monotone mapping (see [1, p. 89—98]).

This paper deals with similar results on explicitly quasiconvex functions and improves some results of our recent paper [12].

1. Introduction. Let X be a real Banach space and let f be a function from X to $\overline{\mathbb{R}}$. Denote by $D(f)$ its *effective domain* $\{x \in X : f(x) < +\infty\}$.

The function f is called *quasiconvex* if

$$f(x + t(y - x)) \leqslant \max\,(f(x), f(y)) \tag{1.1}$$

for all $x, y \in X$ and $t \in [0, 1]$; f is said to be *strictly quasiconvex* if its effective domain is convex and

$$f(x + t(y - x)) < \max\,(f(x),\ f(y)) \tag{1.2}$$

whenever $x, y \in D(f)$, $f(x) \neq f(y)$ and $t \in]0,1[$; f is said to be *explicitly quasiconvex* if it is both quasiconvex and strictly quasiconvex.

The function f is said to be *lower* (*upper*) — *semicontinuous at* x_0 if for each $\lambda < f(x_0)(\lambda > f(x_0))$ there exists a neighbourhood of x_0 such that $\lambda < f(x)$ $(\lambda > f(x))$ whenever x belongs to this neighbourhood; f is *continuous at* x_0 if it is both lower — semicontinuous and upper — semicontinuous at x_0. We say that f is *lower — semicontinuous* if it is lower — semicontinuous at each $x \in X$.

The function f is said to be *hemi-lower* (*upper*) — *semicontinuous at* x if for each $h \in X$ the function $t \to f(x + th)$ from $[0, +\infty[$ to $\overline{\mathbb{R}}$ is lower (upper) — semicontinuous at origin; f is called *hemi-continuous at* x if it is both hemi-lower-semicontinuous and hemi-upper-semicontinuous at x.

We shall denote by L_a and $\overline{L}_a (a \in \overline{\mathbb{R}})$ the *level sets* of f, namely

(1.3) $$L_a = \{x \in X : f(x) < a\},$$

(1.4) $$\overline{L}_a = \{x \in X : f(x) \leqslant a\}.$$

Clearly, f is quasiconvex if and only if each of its level sets is convex. In particular, if f is quasiconvex then the sets $\overline{L}_{-\infty} = \{x \in X : f(x) = -\infty\}$ and $L_{+\infty} = D(f)$ are convex.

Any function f is lower-semicontinuous at each x satisfying $f(x) = -\infty$, is upper-semicontinuous at each x with $f(x) = +\infty$ and is continuous at each x belonging to the interiors of the sets $\{x \in X : f(x) = -\infty\}$ and $\{x \in X : f(x) = +\infty\}$. Also, f is lower-semicontinuous if and only if each of its level sets $\overline{L}_a$ is closed.

Along this paper we shall use the following results:

a) If the function $f : X \to \overline{\mathbb{R}}$ *is strictly quasiconvex and* $L_a \neq \emptyset$, *then* $\overline{L}_a \subset \mathrm{cl}\, L_a$. *If in addition f is lower-semicontinuous, then even the equality* $\overline{L}_a = \mathrm{cl}\, L_a$ *holds* (see [4, Lemma 5]).

b) Any lower-semicontinuous strictly quasiconvex function is quasiconvex and consequently, explicitly quasiconvex (see [6] and [2]).

c) If a lower-semicontinuous quasiconvex function is hemi-upper-semicontinuous at a point x, then it is continuous at x (see [8]).

The function f is said to be *proper* if $f(x) > -\infty$ for every $x \in X$ and $D(f)$ is non-empty.

Denote by X^* the dual space of X and write (x^*, x) instead of $x^*(x)$ for $x \in X$ and $x^* \in X^*$. We shall identify a multivalued mapping $A : X \to 2^{X^*}$ with its graph $A \subset X \times X^*$ and we shall set $D(A) = \{x \in X : Ax \neq \emptyset\}$.

The multivalued mapping $\partial f : X \to 2^{X^*}$, where

(1.5) $$\partial f(x) = \{x^* \in X^* : f(y) \geqslant f(x) + (x^*, y - x) \text{ for all } y \in X\}$$

is called the *subdifferential* of f. Clearly, if f is not the constant $+\infty$, then $D(\partial f)$ is a subset of $D(f)$. The function f is said to be subdifferentiable at x if $x \in D(\partial f)$.

We recall that if f is a lower-semicontinuous proper convex function, then $\mathrm{int}\, D(f) \subset D(\partial f)$ (see [1, Corollary 2.2.1]) and $D(\partial f)$ is a dense subset of $D(f)$ (see [1, Corollary 2.2.2]).

We easily see that the subdifferential of any proper function f is *cyclically monotone*, that is

(1.6) $$(x_0^*, x_1 - x_0) + \ldots + (x_{n-1}^*, x_n - x_{n-1}) + (x_n^*, x_0 - x_n) \leqslant 0$$

for every finite set of pairs $[x_i, x_i^*] \in A$.

We recall that if f is a lower-semicontinuous proper convex function, then its subdifferential ∂f is maximal cyclically monotone (see [1, Theorem 2.2.2]).

2. Generalized subdifferential. Let f be any function from a real Banach space X to $\overline{\mathbb{R}}$. We define the *generalized subdifferential* of f as a

multivalued mapping $F(f)$ from X into X^*, where for each $x \in X$, $F(f)(x)$ is the set of all $x^* \in X^*$ satisfying the following two conditions:

(2.1) $$(x^*, y - x) \geqslant 0 \text{ implies } f(y) \geqslant f(x),$$

(2.2) $$(x^*, y - x) > 0 \text{ implies } f(y) > f(x).$$

This notion is closely related to the *quasi-subdifferential* $\partial^* f$ defined in [5] (see also [14]) by using only condition (2.1) and also to the *tangential* $T(f)$ defined in [3].

Clearly, if f is not the constant function $+\infty$ (i.e., $f \not\equiv +\infty$), then $D(F(f)) \subset D(f)$ and for every $x \in D(F(f))$ the set $F(f)(x) \cup \{0\}$ is a convex cone in X^*(i.e., it is closed with respect to addition and multiplication by non-negative scalars).

Also, $0 \in F(f)(x)$ if and only if $f(y) \geqslant f(x)$ for all $y \in X$ and $F(f)(x) = X^*$ if and only if $f(y) > f(x)$ for all $y \in X, y \neq x$.

The following relation is known between subdifferential and convexity: any function f is convex on each convex subset of $D(\partial f)$. Also, any function f is quasiconvex on each convex subset of $D(\partial^* f)$. A similar relation can be established between generalized subdifferential and explicit quasiconvexity:

PROPOSITION 2.1. *A function f is explicitly quasiconvex on each convex subset of* $D(F(f))$.

Proof. Let C be a convex subset of $D(F(f))$ and let $x, y \in C$ and $t \in [0,1]$. Set $y_t = x + t(y - x)$. Since $x - y_t = -t(y - x)$ and $y - y_t = (1 - t)(y - x)$, we have $(y_t^*, x - y_t) \geqslant 0$ or $(y_t^*, y - y_t) \geqslant 0$, where $y_t^* \in F(f)(y_t)$. If one of these inequalities is strict, then clearly $f(y_t) < \max(f(x), f(y))$. Otherwise, i.e. $(y_t^*, x - y_t) = (y_t^*, y - y_t) = 0$, we have $f(y_t) \leqslant f(x)$ and $f(y_t) \leqslant f(y)$, whence $f(y_t) \leqslant \max(f(x), f(y))$ and if $f(x) \neq f(y)$, then $f(y_t) < \max(f(x), f(y))$. Hence f is explicitly quasiconvex on C.

Let us remark that if f is any proper function then $\partial f \subset F(f)$. In particular, if f is a lower-semicontinuous proper convex function, then $\mathrm{int}\, D(f) \subset F(D(f))$. More generally, we have the following result.

PROPOSITION 2.2. *Let f be an explicitly quasiconvex function from X to* $\mathbb{R}$ *and let* $x \in D(f)$. *If f is upper-semicontinuous at y whenever* $f(y) < f(x)$, *then* $x \in D(F(f))$.

Proof. If x is a minimum point of f, then $0 \in F(f)(x)$ and so $x \in D(F(f))$.

Next let us suppose that f does not achieve its minimum at x. Then the level set $L_{f(x)}$ is non-empty and since f is quasiconvex, it is also convex. Moreover, since f is upper-semicontinuous at each $y \in L_{f(x)}$, we have that $L_{f(x)}$ is open. Then, since $x \notin L_{f(x)}$, there exists a non-trivial $x^* \in X^*$ such that $(x^*, y - x) < 0$ for all $y \in L_{f(x)}$(see [1, Theorem 1.1.9]). Hence $(x^*, y - x) \geqslant 0$ implies $f(y) \geqslant f(x)$. Thus, x^* satisfies condition (2.1).

Clearly, $(x^*, y - x) \leqslant 0$ for all $y \in \mathrm{cl}\, L_{f(x)}$. In particular, this inequality holds for every $y \in \overline{L}_{f(x)}$ because $\overline{L}_{f(x)} \subset \mathrm{cl}\, L_{f(x)}$. It follows that $(x^*, y - x) > 0$ implies $f(y) > f(x)$. Thus, x^* also satisfies condition (2.2).

Therefore, $x^* \in F(f)(x)$ and hence $x \in D(F(f))$, which completes the proof.

COROLLARY 2.3. *Let f be a lower-semicontinuous strictly quasiconvex function from X to $\overline{\mathbb{R}}$. If f is hemi-upper-semicontinuous at each interior point of $D(f)$, then*

$$\text{int}\, D(f) \subset D(F(f)). \tag{2.3}$$

Proof. In our assumptions the function f is explicitly quasiconvex and continuous at each interior point of $D(f)$. Next we may apply Proposition 2.2 to the function $\tilde{f}: X \to \overline{\mathbb{R}}$ with $D(\tilde{f}) = \text{int}\, D(f)$, $\tilde{f}(x) = f(x)$ for all $x \in \text{int}\, D(f)$. Finally, we use the fact that

$$F(\tilde{f})(x) = F(f)(x)$$

for each $x \in \text{int}\, D(f)$. Indeed, let $x \in \text{int}\, D(f)$. If $x^* \in F(f)(x)$ and $(x^*, y - x) \geq 0$ (>0), then $f(y) \geq f(x)$, $(f(y) > f(x))$ and since $\tilde{f}(x) = f(x)$ and $\tilde{f} \geq f$, we get $\tilde{f}(y) \geq \tilde{f}(x)$, $(\tilde{f}(y) > \tilde{f}(x))$, which shows that $x^* \in F(\tilde{f})(x)$. Hence $F(f)(x) \subset F(\tilde{f})(x)$. Now let $x^* \in F(\tilde{f})(x)$ and suppose that $(x^*, y - x) \geq 0$ (>0). If $y \in \text{int}\, D(f)$, then obviously $f(y) \geq f(x)$ $(f(y) > f(x))$. Let y be a boundary point of $D(f)$. Then $y_t = x + t(y - x) \in \text{int}\, D(f)$ and $(x^*, y_t - x) \geq 0$ (>0) for all $t \in]0,1[$. It follows that $f(y_t) \geq f(x)$ $(f(y_t) > f(x))$. Whence, since f is strictly quasiconvex, we deduce that $f(y) \geq f(x) (f(y) > f(x))$. Hence $x^* \in F(f)(x)$ and therefore $F(\tilde{f})(x) \subset F(f)(x)$. The proof is thus complete.

PROPOSITION 2.4. *Let $f: X \to \overline{\mathbb{R}}$ be a quasiconvex function. If f is Gâteaux differentiable at x, $\text{grad}\, f(x) \neq 0$ and f is upper-semicontinuous at each y satisfying $f(y) < f(x)$, then $x \in D(F(f))$ and $\text{grad}\, f(x) \in F(f)(x)$.*

Proof. We shall prove that $\text{grad}\, f(x) \in F(f)(x)$.

Assume that $(\text{grad}\, f(x), y - x) \geq 0$ but nevertheless $f(y) < f(x)$. Then, since $\text{grad}\, f(x) \neq 0$ and f is upper-semicontinuous at y, we can find $y_1 \in D(f)$ such that $(\text{grad}\, f(x), y_1 - x) > 0$ and $f(y_1) < f(x)$. By $(\text{grad}\, f(x), y_1 - x) > 0$ we get $f(x + t(y_1 - x)) > f(x)$ for all $t \in]0, t_0]$ (t_0 being a certain number ≤ 1), which contradicts the quasiconvexity of f. This shows that $\text{grad}\, f(x)$ satisfies condition (2.1).

Now suppose that $(\text{grad}\, f(x), y - x) > 0$. Then $f(x + t(y - x)) > f(x)$ for all $t \in]0, t_0] (t_0 \leq 1)$. On the other hand, as we have already proved $f(y) \geq f(x)$ and in consequence, by the quasiconvexity of f, we must have $f(y) \geq f(x + t(y - x))$ for all $t \in [0, 1]$. It follows that $f(y) > f(x)$. Thus, $\text{grad}\, f(x)$ also satisfies condition (2.2).

Therefore, $\text{grad}\, f(x) \in F(f)(x)$, as claimed.

COROLLARY 2.5. *Let $f: X \to \overline{\mathbb{R}}$ be a quasiconvex function. If f is upper-semicontinuous and Gâteaux differentiable at each point of $D(f)$ and $\text{grad}\, f(x) \neq 0$ for all $x \in D(f)$, then*

$$\text{grad}\, f \subset F(f). \tag{2.4}$$

In what follows we shall determine the generalized subdifferential of a strictly quasiconvex function at any point of its effective domain.

Let us set

$$K(x) = \{h \in X : \text{there is } t > 0 \text{ such that } f(x + th) < f(x)\}$$

$$\overline{K}(x) = \{h \in X : \text{there is } t > 0 \text{ such that } f(x + th) \leq f(x)\}$$

and denote

$$K'(x) = \{x^* \in X^* : (x^*, h) < 0 \text{ for all } h \in K(x)\}$$

$$\overline{K}^0(x) = \{x^* \in X^* : (x^*, h) \leq 0 \text{ for all } h \in \overline{K}(x)\}.$$

PROPOSITION 2.6. *Let $f: X \to \overline{\mathbb{R}}$ be any function.*

(i) *If f achieves its minimum at x, then*

$$F(f)(x) = \overline{K}^0(x). \tag{2.5}$$

(ii) *If f is strictly quasiconvex and $x \in D(f)$ is not a minimum point of f, then*

$$F(f)(x) = K'(x). \tag{2.6}$$

Proof. (i) Let $x^* \in F(f)(x)$ and let h be any element of $\overline{K}(x)$. Then we have $(x^*, h) \leq 0$. Indeed, otherwise (i.e., $(x^*, h) > 0$) the inequality $(x^*, (x + th) - x) > 0$ implies that $f(x + th) > f(x)$ for all $t > 0$, which is absurd because $h \in \overline{K}(x)$. Therefore $(x^*, h) \leq 0$ for all $h \in \overline{K}(x)$, hence $F(f)(x) \subset \overline{K}^0(x)$.

Now let $x^* \in \overline{K}^0(x)$. Since x is a minimum point of f, obviously x^* satisfies condition (2.1). To verify (2.2) let us assume that $(x^*, y - x) > 0$ and that, nevertheless, $f(y) = f(x)$. Then $y - x \in \overline{K}(x)$ and in consequence $(x^*, y - x) \leq 0$, which is absurd. Thus, x^* satisfies (2.2). Therefore $x^* \in F(f)(x)$ and so $\overline{K}^0(x) \subset F(f)(x)$.

(ii) Let $x^* \in F(f)(x)$ and let h be any element of $K(x)$. We want to show that $(x^*, h) < 0$. Indeed, in the opposed case, from $(x^*, (x + th) - x) \geq 0$ we derive $f(x + th) \geq f(x)$ for all $t > 0$, which is absurd because $h \in K(x)$. Hence $F(f)(x) \subset K'(x)$.

To prove the converse inclusion let us consider an arbitrary $x^* \in K'(x)$. First let y be such that $(x^*, y - x) \geq 0$. Then $y - x \notin K(x)$. Hence $f(y) \geq f(x)$. Thus, x^* satisfies (2.1).

Next, let y be such that $(x^*, y - x) > 0$. Then there exists a neighbourhood V of y such that $(x^*, y_1 - x) > 0$ for all $y_1 \in V$ whence, by what has just proved, $f(y_1) \geq f(x)$ for all $y_1 \in V$. It follows that $y \notin \text{cl}\, L_{f(x)}$. On the other hand, since f is strictly quasiconvex and $L_{f(x)}$ is non-empty, we have $\overline{L}_{f(x)} \subset \text{cl}\, L_{f(x)}$. Thus, $y \notin \overline{L}_{f(x)}$ and hence $f(y) > f(x)$. This shows that x^* satisfies (2.2).

It follows that $x^* \in F(f)(x)$ and hence $K'(x) \subset F(f)(x)$. This completes the proof.

Remark 2.7. *Let $f: X \to \overline{\mathbb{R}}$ be a strictly quasiconvex function. If f does not achieve its minimum at x and f is upper-semicontinuous at each point of $L_{f(x)}$, then*

$$K'(x) = \overline{K}^0(x) \setminus \{0\}. \tag{2.7}$$

Indeed, let $x^* \in K'(x)$ and $h \in \bar{K}(x)$. We want to show that $(x^*, h) \leqslant 0$. Since this inequality trivially holds if h also belongs to $K(x)$, next let us suppose that $h \in \bar{K}(x) \setminus K(x)$, i.e. $f(x + th) = f(x)$ for a certain $t > 0$. Then, by $x + th \in \bar{L}_{f(x)} \subset \mathrm{cl}\, L_{f(x)}$, it follows that each neighbourhood of h contains at least one point h_1 such that $f(x + th_1) < f(x)$, whence $(x^*, h_1) < 0$. In consequence $(x^*, h) \leqslant 0$, as desired. Thus, $K'(x) \subset \bar{K}^0(x) \setminus \{0\}$.

Conversely, let $x^* \in \bar{K}^0(x)$, $x^* \neq 0$ and let $h \in K(x)$. Then, there exists $t > 0$ such that $f(x + th) < f(x)$ and also $(x^*, h) \leqslant 0$. We shall show that the last inequality strictly holds. Indeed, otherwise, since $x^* \neq 0$ and f is upper-semicontinuous at $x + th$, there exists h_1 such that $f(x + th_1) < f(x)$ and $(x^*, h_1) > 0$. Hence $h_1 \in \bar{K}(x)$ while $(x^*, h_1) > 0$, which is a contradiction. Therefore $(x^*, h) < 0$ and so $x^* \in K'(x)$. This shows that $\bar{K}^0(x) \setminus \{0\} \subset K'(x)$, thereby completing the proof of (2.7).

COROLLARY 2.8. *Let $f : X \to \bar{\mathbb{R}}$ be an explicitly quasiconvex function. If f is Gâteaux differentiable at x, $\mathrm{grad}\, f(x) \neq 0$ and f is upper-semicontinuous at each y such that $f(y) < f(x)$, then*

(2.8) $$F(f)(x) = \{t\, \mathrm{grad}\, f(x) : t > 0\}.$$

Proof. By Proposition 2.4, $\{t\ \mathrm{grad}\, f(x) : t > 0\} \subset F(f)(x)$. For the converse inclusion let us remark that

$$\{h \in X : (\mathrm{grad}\, f(x), h) < 0\} \subset K(x),$$

whence $K'(x) \subset \{t\, \mathrm{grad}\, f(x) : t > 0\}$. Next, equality (2.8) follows by Proposition 2.6 (ii).

COROLLARY 2.9. *Let $f : X \to \bar{\mathbb{R}}$ be a lower-semicontinuous proper convex function.*

(i) *If $x \in D(\partial f)$ and f does not achieve its minimum at x, then*

(2.9) $$\mathrm{cl}\, F(f)(x) = \mathrm{cl}\, (\cup \{\lambda f(x) : \lambda > 0\})$$

(ii) *If in addition $x \in \mathrm{int}\, D(f)$, then*

(2.10) $$F(f)(x) = \cup \{\lambda \partial f(x) : \lambda > 0\}.$$

Proof. (i) By Proposition 2.6 (ii), $F(f)(x) = K'(x)$; clearly, $K'(x) \subset \subset K^0(x)$ and since $\bar{K}(x) \subset \mathrm{cl}\, K(x)$, we have $\bar{K}^0(x) \subset \bar{K}^0(x)$. Next $\bar{K}^0(x) = \mathrm{cl}(\cup\{\lambda\partial f(x) : \lambda > 0\})$ (see [13, Theorem 23.7]).

(ii) Since $\partial f(x)$ is weakly* closed and bounded (because $x \in \mathrm{int}\, D(f)$), we have that $\partial f(x)$ is weakly* compact. Consequently, the convex cone $\cup\{\lambda\partial f(x) : \lambda \geqslant 0\}$ generated by $\partial f(x)$, is closed. Thus, $\bar{K}^0(x) \setminus \{0\} = \cup\{\lambda\partial f(x) : \lambda > 0\}$.

3. Pseudomonotonicity of generalized subdifferential. Let A be a multivalued mapping from X into X^*.

We say that A is *pseudomonotone* if for every $x, y \in D(A)$, the following condition is fulfilled:

(3.1) $(y^*, y - x) \geqslant 0$ for all $y^* \in Ay$, whenever there exists $x^* \in Ax$ such that $(x^*, y - x) \geqslant 0$.

Clearly, any monotone mapping is pseudomonotone.

If the mapping A is univoque, then condition (3.1) reduces to

(3.2) $$(Ax, y - x) \geqslant 0 \text{ implies } (Ay, y - x) \geqslant 0,$$

which is just the condition of the pseudomonotonicity defined in [7].

We say that A is *cyclically pseudomonotone* if

(3.3) $$\min\, ((x_0^*, x_1 - x_0), \ldots, (x_{n-1}^*, x_n - x_{n-1}), (x_n^*, x_0 - x_n)) < 0$$
$$\text{or } (x_0^*, x_1 - x_0) = \ldots = (x_{n-1}^*, x_n - x_{n-1}) = (x_n^*, x_0 - x_n) = 0,$$

for every finite set of pairs $[x_i, x_i^*] \in A$.

Obviously, any cyclically monotone mapping is cyclically pseudomonotone.

A (cyclically) pseudomonotone mapping $A \subset X \times X^*$ is said to be *maximal (cyclically) pseudomonotone with respect to C* (where $C \subset X$), provided that if $B \subset X \times X^*$ is a (cyclically) pseudomonotone mapping such that $Ax \subset Bx$ for all $x \in X$ and $Ax = Bx$ for all $x \in X \setminus C$, then $A = B$.

PROPOSITION 3.1. *The following statements are equivalent:*

1°. *The mapping A is pseudomonotone;*

2° *For every $x, y \in D(A)$, we have*

(3.4) *$(y^*, y - x) > 0$ for all $y^* \in Ay$, whenever there exists $x^* \in Ax$ such that $(x^*, y - x) > 0$;*

3°. *We have*

(3.5) $$\min\, ((x^*, y - x), (y^*, x - y)) < 0$$
$$\text{or } (x^*, y - x) = (y^*, x - y) = 0,$$

for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$.

The proof of Proposition 3.1 can be found in [12].

By Proposition 3.1.3° we immediately see that any cyclically pseudomonotone mapping is pseudomonotone.

PROPOSITION 3.2. *The generalized subdifferential $F(f)$ of any function f is cyclically pseudomonotone.*

The proof is immediate and can be found in [12].

PROPOSITION 3.3. *Let C be a non-empty open convex subset of X and let $f : C \to \mathbb{R}$ be Gâteaux differentiable on C. In order that $\mathrm{grad}\ f \subset F(f)$, it is necessary and sufficient that $\mathrm{grad}\, f$ be pseudomonotone.*

Proof. The necessity of this condition is immediate because, by Proposition 3.2, $F(f)$ is pseudomonotone.

To prove the sufficiency, let us assume that $\mathrm{grad}\, f$ is pseudomonotone. Let x and y be two arbitrary points of C such that $(\mathrm{grad}\, f(x), y - x) \geqslant 0$. Then, $(\mathrm{grad}\ f(x), (x + t(y - x)) - x) \geqslant 0$ for every $t \in [0, 1]$. Since $\mathrm{grad}\ f$ is pseudomonotone, we may infer that $(\mathrm{grad}\ f(x + t(y - x)), y - x) \geqslant 0$ for all $t \in [0, 1]$. It follows that the function $g : [0, 1] \to \mathbb{R}$, $g(t) = f(x + t(y - x))$ $(t \in [0, 1])$, is non-decreasing on $[0, 1]$. Therefore

$g(1) = f(y) \geqslant f(x) = g(0)$. Similarly we can prove that $(\text{grad}\, f(x), y - x) > 0$ implies $f(y) > f(x)$. Thus, $\text{grad} f(x) \in F(f)(x)$ and the proof is complete.

Remark 3.4. Let C be a non-empty open convex subset of X and $f : C \to \mathbb{R}$ be Gâteaux differentiable on C. According to Proposition 2.1., a necessary condition that $\text{grad} f \subset F(f)$ is that f be explicitly quasi-convex. A sufficient condition that $\text{graf} f \subset F(f)$ is that f be quasiconvex, upper-semicontinuous and $\text{grad} f(x) \neq 0$ for all $x \in C$ (see Proposition 2.3). The necessary and sufficient condition that $\text{grad} f \subset F(f)$ is that f be pseudoconvex (see [7, Theorem 3.1]).

Next we shall prove a theorem on the maximal pseudomonotonicity of the generalized subdifferential.

PROPOSITION 3.5. *Let f be a function from X to $\overline{\mathbb{R}}$ and let C a non--empty subset of $D(F(f))$ such that*

(3.6) *if $x \in C$ and $x + h \in \overline{L}_{f(x)}$, then $x + th \in D(F(f))$ for $0 \leqslant t < t_0$ (t_0 depending on x and h).*

If f is lower-semicontinuous at each $x \in C$, then $F(f)$ is maximal pseudomonotone with respect to C.

Proof. Let $x \in C$ and $x^* \in X^*$. Suppose that the mapping $F(f) \cup \cup \{[x, x^*]\}$ is pseudomonotone. Then

$$\min((x^*, y - x), (y^*, x - y)) < 0$$
(3.7)
$$\text{or } (x^*, y - x) = (y^*, x - y) = 0$$

for all $[y, y^*] \in F(f)$.

We shall prove that (3.7) assures that

(3.8) $$(x^*, h) > 0 \text{ implies } f(x + h) > f(x)$$

and

(3.9) $$(x^*, h) \geqslant 0 \text{ implies } f(x + h) \geqslant f(x).$$

To this end, let us first consider an arbitrary h such that $(x^*, h) > 0$. Suppose that, nevertheless, $f(x + h) \leqslant f(x)$. Hence $x + h \in \overline{L}_{f(x)}$. By (3.6) there exists $t_0 > 0$ such that $x + th \in D(F(f))$ for $0 \leqslant t < t_0$. Clearly, we may assume that $t_0 \leqslant 1$. Applying (3.7) to $y = y_t = x + th$ $(0 < t < t_0)$ and taking into account that $(x^*, y_t - x) = t(x^*, h) > 0$, we get $(y_t^*, h) > 0$. Since $y_t^* \in F(f)(y_t)$, by $(y_t^*, (y_t + sh) - y_t) > 0$, we must have $f(x + (t + s)h) > f(x + th)$ for all $0 < t < t_0$ and $s > 0$. In particular, for $s = 1 - t$ we have $f(x + h) > f(x + th)$ $(0 < t < t_0)$ and for $s = t$ we have $f(x + 2th) > f(x + th)$ $(0 < t < t_0)$. Whence

$$f(x) \geqslant f(x + h) > f(x + th) > \ldots > f\left(x + \frac{t}{2^n} h\right) > \ldots,$$

which is impossible due to the lower-semicontinuity of f at x. Hence (3.8) must hold.

Similarly, if $(x^*, h) = 0$ and we suppose that $f(x + h) < f(x)$, we get

$$f(x) > f(x + h) \geqslant f(x + th) \geqslant \ldots \geqslant f\left(x + \frac{t}{2^n} h\right) \geqslant \ldots,$$

which once again contradicts the lower-semicontinuity of f at x. Hence (3.9) must hold too.

Therefore $x^* \in F(f)(x)$, thereby proving the maximal pseudomonotonicity of $F(f)$ with respect to C.

Remark 3.6. *Under the assumptions of Proposition 3.5, $F(f)$ is maximal cyclically pseudomonotone with respect to C.*

Indeed, it is easy to see that the maximal pseudomonotonicity with respect to C of a cyclically pseudomonotone mapping implies its maximal cyclically pseudomonotonicity with respect to C.

COROLLARY 3.7. *Let $f : X \to \overline{\mathbb{R}}$ be a lower-semicontinuous quasiconvex function. If $D(F(f)) = D(f)$, then the generalized subdifferential $F(f)$ is maximal cyclically pseudomonotone with respect to $D(f)$.*

Proof. Apply Proposition 3.5, where $C = D(f)$.

COROLLARY 3.8. *Let $f : X \to \overline{\mathbb{R}}$ be a lower-semicontinuous strictly quasiconvex function having $\text{int}\, D(f) \neq \emptyset$. If f is hemi-upper-semicontinuous at each interior point of $D(f)$, then the generalized subdifferential $F(f)$ is maximal cyclically pseudomonotone with respect to $\text{int}\, D(f)$.*

Proof. Use Corollary 2.3 and apply Proposition 3.5 with $C = \text{int}\, D(f)$.

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