MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 17, N° 1, 1988, pp. 63—64

NORM-ONE PROJECTIONS ON SOME FUNCTION SPACES

I. RAŞA
(Cluj-Napoca)

1. Let X be a compact Hausdorff space and let $C(X)$ be the space of all real-valued continuous functions on X endowed with sup-norm. Let V and W be linear subspaces of $C(X)$ such that $1 \in V \subset W$. Denote by $\mathrm{Prob}(X)$ the set of all probability Radon measures on X; let δ_x be the Dirac measure at $x \in X$.

As in [2], define the Choquet boundary of V with respect to W by $\mathrm{Ch}_W(V) = \{x \in X : \text{if } \mu \in \mathrm{Prob}(X) \text{ and } \mu = \delta_x \text{ on } V, \text{ then}$
$\mu = \delta_x \text{ on } W\}$.

Then $\mathrm{Ch}(V) := \mathrm{Ch}_{C(X)}(V)$ is the usual Choquet boundary of V. If W is the uniqueness closure of V, i.e., $W = \{f \in C(X) : \mu(f) = f(x)$ for all $x \in X$ and all $\mu \in \mathrm{Prob}(X)$ such that $\mu = \delta_x$ on $V\}$, then $\mathrm{Ch}_W(V) = X$.

The following result is a consequence of Theorem 2 of C. Franchetti [2] (see also E. Scheffold [3], Lemma 2).

THEOREM 1. *Let* $Y \subset \mathrm{Ch}_W(V)$ *and* $L \in (W, C(X))'$, $\| L \| = 1$. *If* $\mathrm{L}p = p$ *on* Y *for all* $p \in V$, *then* $\mathrm{L}f = f$ *on* Y *for all* $f \in W$.

We shall give a direct proof of this theorem.

Let $y \in Y$. Let $\nu(f) = \mathrm{L}f(y)$, $f \in W$. Then $\nu \in W'$ and $\|\nu\| \leq 1$. Since $1 \in V$, we have $\nu(1) = 1$, hence $\|\nu\| = 1$. Using the Hahn-Banach theorem we find a $\mu \in C(X)'$ with $\mu = \nu$ on W and $\|\mu\| = 1$. Then $\mu(1) = \nu(1) = 1$, hence $\|\mu\| = \mu(1) = 1$. It follows that $\mu \in \mathrm{Prob}(X)$.

For $p \in V$ we have $\mu(p) = \nu(p) = \mathrm{L}p(y) = p(y)$, hence $\mu = \delta_y$ on V. Since $y \in \mathrm{Ch}_W(V)$, we have $\mu = \delta_y$ on W. Then $\nu = \delta_y$ on W, i.e., $\mathrm{L}f(y) = \nu(f) = f(y)$ for all $f \in W$.

THEOREM 2. *Suppose that* W *separates* X *and* $\mathrm{Ch}(V) = \mathrm{Ch}(W)$. *If* $L \in (W, W)'$, $\|L\| = 1$ *and* $\mathrm{L}p = p$ *on* $\mathrm{Ch}(W)$ *for all* $p \in V$, *then* L *is the identity operator on* W.

Proof. Let $f \in W$. Since $\mathrm{Ch}(W) = \mathrm{Ch}(V) \subset \mathrm{Ch}_W(V)$, from Theorem 1 it follows that $\mathrm{L}f - f = 0$ on $\mathrm{Ch}(W)$. Moreover, $\mathrm{L}f - f \in W$; by using the Bauer's maximum principle we deduce that $\mathrm{L}f - f = 0$ on X.

The following result was obtained, for subspaces of $C_c(X)$, by E. Briem ([1], Th. 7).

COROLLARY 1. *Suppose that* W *separates* X *and* $\mathrm{Ch}(V) = \mathrm{Ch}(W)$. *If* $L \in (W, V)'$ *is a norm-one projection, then* $V = W$ *and* L *is the identity operator.*

Proof. It suffices to apply Theorem 2.

2. In [4], D. E. Wulbert has proved :

PROPOSITION 1. *Let X be a compact Hausdorff space which contains at most a finite number of isolated points. Let P be a subspace of finite codimension in $C(X)$ and let L be a norm-one linear operator defined on $C(X)$ which acts as the identity on P. Then for each f in $C(X)$ and each nonisolated y in X, $\mathrm{Lf}(y) = f(y)$.*

Using Theorem 1, the following related result can be proved.

PROPOSITION 2. *Let X be a compact Hausdorff space. Let P be a subspace of finite codimension in $C(X)$ which contains the constant functions and separates the points of X. Let $L \in (C(X), C(X))'$, $\|L\| = 1$, $\mathrm{Lp} = p$ on $\mathrm{Ch}(P)$ for all $p \in P$. Then $\mathrm{Lf}(y) = f(y)$ for each $f \in C(X)$ and each nonisolated y in X.*

Proof. We apply Theorem 1 with $V = P$, $W = C(X)$ and $Y = \mathrm{Ch}(P)$. It follows that $\mathrm{Lf} = f$ on $\mathrm{Ch}(P)$ for all $f \in C(X)$. Now it suffices to show that each nonisolated point of X is in the closure of $\mathrm{Ch}(P)$.

Suppose the contrary. Then there exist a nonisolated point y and an open set U which contains y and does not intersect $\mathrm{Ch}(P)$. Let $m = \mathrm{codim}(P)$. As in the Wulbert's proof of Proposition 1 (see [4], p. 389) we construct $m + 1$ continuous functions, all of norm one, which have disjoint supports and such that the support of each is contained in U. Then there exists a linear combination $f \neq 0$ of these functions which is in P. But f vanishes on $\mathrm{Ch}(P)$, hence by the Bauer's maximum principle f vanishes on X, a contradiction.

REFERENCES

[1]. Briem, E., *Korovkin type theorems for subspaces of* $C_C(X)$, Kobenhavns Universitet, Mat. Inst., Preprint Series, No. 11, 1979.

[2]. Franchetti, C., *Convergenza di operatori in sollospazi dello spazio* $C(Q)$, Boll.Un.Mat. Ital., IV (1970), 668–676.

[3]. Scheffold, E., *Ein allgemeiner Korovkin-Satz für lokalkonvexe Vektorverbände*, Math. Z., 132 (1973), 209–214.

[4]. Wulbert, D. E., *Convergence of operators and Korovkin's theorem*, J. Approx. Theory, 1 (1968). 381–390.

Received 2.IX.1987

Institutul Politehnic
Str. Emil Isac, 15
3400 Cluj-Napoca
România