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A PROBABILISTIC APPROACH TO THE INTEGRABILITY CONCEPT

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The purpose of this paper is to develop an integration theory for probabilistic functions, with respect to a positive measure, using probabilistic submeasures families and the topological ring of sets associated to them. Since τ_M is the unique triangle function which satisfies the functional equation

$$\tau[F(j/a), F(j/b)] = F(j/a + b),$$

in this paper we will consider the commutative semigroup $(\mathscr{R}^+, \tau_M)$.

1. Preliminary concepts. Let $\mathscr{R}^+$ be the set of all probability distribution functions of non-negative random variables, i.e.

$$\mathscr{R}^+ = \{F;\ F : [-\infty, \infty] \to [0, 1],\ F(0) = 0, F(\infty) = 1, F \text{ is non-decreasing and left-continuous on } [-\infty, \infty]\}$$

and let H_0 be the distribution function in $\mathscr{R}^+$ defined by $H_0(x) = 0$ for $x \leqslant 0$ and $H_0(x) = 1$ for $x > 0$. The mapping $\tau_M : \mathscr{R}^+ \times \mathscr{R}^+ \to \mathscr{R}^+$ defined by

$$\tau_M(F, G)(x) = \sup_{u+v=x} \text{Min}\,(F(u), G(v))$$

is a triangular function, and $(\mathscr{R}^+, \tau_M)$ is a commutative semigroup with unit H_0. We denote by $(\mathscr{R}^+, \tau_M, d_{\mathscr{L}})$ the uniform topological semigroup with respect to the modified Levy's metric $d_{\mathscr{L}}$[2].

If j denotes the identity function on $[-\infty, \infty]$, then for any F in $\mathscr{R}^+$ and any $a > 0$, the distribution function on $\mathscr{R}^+$, whose value for any $x \geqslant 0$ is $F(x/a)$ may be conveniently denoted by $F(j/a)$. In [8] the authors showed that:

$$\tau_M(F(j/a), F(j/b)) = F(j/a + b);\ F \in \mathscr{R}^+,\ a, b > 0.$$

In this paper we will make the convention $F(j/0) = H_0, F(j/\infty) = H_0$, and we write

$$\overset{n}{\underset{i=1}{\tau_M}}(F_i) = \tau_M(F_1, \tau_M(F_2, \ldots, \tau_M(F_{n-1}, F_n), \ldots)$$

Let T be a t-norm, S a nonempty set and $\mathscr{A} \subset \mathscr{P}(S)$ an algebra of subsets. We say that a mapping $\gamma : \mathscr{A} \to \mathscr{R}^+$ is a probabilistic submeasure on $\mathscr{A}$ with respect to T if

(i) $\gamma_\Phi = H_0$; (ii) $A \subset B$ implies $\gamma_A(x) \geqslant \gamma_B(x)$, $x > 0$, $A, B \in \mathscr{A}$

(iii) for each $A, B \in \mathscr{A}$, $\gamma_{A \cup B}(x+y) \geqslant T(\gamma_A(x), \gamma_B(x))$, $x > 0$, $y > 0$

If $\gamma : \mathscr{A} \to \mathscr{R}^+$ is a probabilistic submeasure with respect to a continuous t-norm, the outer submeasure induced by γ, denoted by $\gamma^* : \mathscr{P}(S) \to \mathscr{R}^+$, [1], where: $\gamma^*_A(x) = \sup\{\gamma_E(x); A \subseteq E \in \mathscr{A}\}$, $A \subset S$.

Let $\Gamma = \{\Gamma_i\}_{i \in I}$ be a non empty family of probabilistic submeasures on $\mathscr{A}$ with the t-norms T_i, $i \in I$ continuous. Let there $\mathbb{B}_\Gamma = \{\mathscr{V}_{K,\varepsilon,\lambda};$ K finite $\subset I$, $\varepsilon > 0$, $\lambda > 0\}$, where $\mathscr{V}_{K,\varepsilon,\lambda} = \{A \in \mathscr{P}(S);\ \gamma^{i*}_A(\varepsilon) > 1 - \lambda,$ $i \in K\}$. By [4] there exists a unique topology $\mathscr{T}_\Gamma$ on $\mathscr{P}(S)$, so that $[\mathscr{P}(S)](\Gamma) = (\mathscr{P}(S), \Delta, \cap, \mathscr{T}_\Gamma)$ is a topological ring, and $\mathbb{B}_\Gamma$ is a normal base of neighbourhoods of $\emptyset$ for this topology. We denote by $E_\alpha \xrightarrow{\Gamma} E$, the convergence from $[\mathscr{P}(S)](\Gamma)$. We say that the set $E \in \mathscr{A}$ is Γ-finite (Γ-negligible) if for any $i \in I$, $\sup\limits_{x>0} \gamma^i_E(x) = 1$ (respectively $\gamma^i_E(x) = H_0(x)$, $x > 0$).

In the sequel we will define the integrability of functions from $[\mathscr{R}^+]^S$. To K finite $\subset I$, $\varepsilon > 0$, $\alpha > 0$, $\lambda > 0$ we will associate the set:

$$\mathscr{W}_K(\varepsilon, \lambda; \alpha) = \{(f, g) \in \mathscr{R}^+ \times \mathscr{R}^+;\ \gamma^{i*}_{\{s \in S;\ d_{\mathscr{L}}(f(s),\ g(s)) \geqslant \alpha\}}(\varepsilon) > 1 - \lambda\}$$

The family $\{\mathscr{W}_K(\varepsilon, \lambda, \alpha)\}_{\varepsilon>0,\ \lambda>0,\ \alpha>0}$ forms a fundamental system of entourages for a uniform structure $\mathbb{U}_\Gamma$ on $[\mathscr{R}^+]^S$.

We write $[\mathscr{R}^+]^S(\Gamma) = ([\mathscr{R}^+]^S, \mathbb{U}_\Gamma)$. For $f \in [\mathscr{R}^+]^S$ we write $f(s) = F_s$, $s \in S$. If $\{f_\alpha\}$ is a generalized sequence from $[\mathscr{R}^+]^S$ and if $\{f_\alpha\}$ converges to f in $[\mathscr{R}^+]^S(\Gamma)$ we say that $\{f_\alpha\}$ converges to f in Γ-submeasures and we write $f_\alpha \xrightarrow{\Gamma} f$.

We will fix a positive measure $\mu : \mathscr{A} \to [0, \infty)$ and we will choose a family of probabilistic submeasures Γ_μ so that the following continuity axioms are satisfied:

C_1) For every $A \in \mathscr{A}$, and every entourage $\mathscr{W}$ from $\mathscr{R}^+$ there exists the entourage $\mathscr{U}$ from $\mathscr{R}^+$ with the following property: for any $n \in N$, if $\{(F^i, G^i)\}_{i=1}^n$ is from $\mathscr{U}$ and if $\{E_i\}_{i=1}^n$ is a sequence of sets disjoint two by two from $\mathscr{A}$, then:

$$\left(\overset{n}{\underset{i=1}{\tau_M}} [F^i(j/\mu(E_i \cap A))],\ \overset{n}{\underset{i=1}{\tau_M}} [G^i(j/\mu(E_i \cap A))]\right) \in \mathscr{W}$$

C_2) For any $F \in \mathscr{R}^+$, $\lim\limits_{\substack{E \xrightarrow{\Gamma_\mu} \Phi \\ E \in \mathscr{A}}} F(j/\mu(E)) = H_0$

2. The integrability of the functions from $[\mathscr{R}^+]^S$.

DEFINITION 1. It is said that a step function $f \in [\mathscr{R}^+]^S$ is Γ_μ-integrable if

(i) f takes a finite number of distinct values $F^1, F^2, \ldots, F^n \in \mathscr{R}^+$ on the sets $E_1, E_2, \ldots, E_n \in \mathscr{A}$ respectively.

(ii) for $i = 1, 2, 3, \ldots, n$, $F^i = H_0$, it follows that E_i is Γ_μ-finite (if $F^i = H_\infty$ it results that E_i is Γ_μ-negligible)

For $E \in \mathscr{A}$, the Γ_μ-integral of f on E by definition is

$$\int_E f\,\mathrm{d}\mu = \overset{n}{\underset{i=1}{\tau_M}} F^i(j/\mu(E_i \cap E))$$

We denote by $\mathscr{E}(\Gamma_\mu)$ the Γ_μ-integrable step functions set.

THEOREM 1.

(i) *Relatively to the operation* τ_M *defined in* $\mathscr{E}(\Gamma_\mu)$ *by* $\tau_M(f, g)_s = \tau_M(F_s, G_s)$, *the space* $\mathscr{E}(\Gamma_\mu)$ *is a semigroup of* $[\mathscr{R}^+]^S$.

(ii) *For* $E \in \mathscr{A}$, *the map* $f \to \int_E f\,\mathrm{d}\mu$ *of* $\mathscr{E}(\Gamma_\mu)$ *in* $\mathscr{R}^+$ *is additive*:

$$\int_E \tau_M(f, g)\,\mathrm{d}\mu = \tau_M\left(\int_E f\,\mathrm{d}\mu, \int_E g\,\mathrm{d}\mu\right)\ f, g \in \mathscr{E}(\Gamma_\mu).$$

(iii) *For* $f \in \mathscr{E}(\Gamma_\mu)$, *the map* $E \to \nu(E)$, $E \in \mathscr{A}$, $\nu_E = \nu(E) = \int_E f\,\mathrm{d}\mu$ *is additive*:

$$\nu_{\bigcup\limits_{i=1}^n E_i} = \overset{n}{\underset{i=1}{\tau_M}} (\nu E_i),\ E_i \cap E_j = \Phi,\ i \neq j,\ \textit{and}\ \nu_\Phi = H_0$$

(iv) *For* $f \in \mathscr{E}(\Gamma_\mu)$, $\lim\limits_{E \xrightarrow{\Gamma_\mu} \Phi} \nu_E = \lim\limits_{E \xrightarrow{\Gamma_\mu} \Phi} \int_E f\,\mathrm{d}\mu = H_0$

The proof follows from definition 1 and axiom C_1.

The extension of the integral from the step functions to the arbitrary functions from $[\mathscr{R}^+]^S$ is based on the following result:

LEMMA 1. *Let* $\{f_\alpha\}$ *be a generalized sequence from* $\mathscr{E}(\Gamma_\mu)$ *which is Cauchy in* $[\mathscr{R}^+]^S(\Gamma_\mu)$. $\left\{\int_E f_\alpha\,\mathrm{d}\mu\right\}$ *in order to be a Cauchy sequence in* $\mathscr{R}^+$ *uniform with respect to* $E \in \mathscr{A}$, *it is necessary and sufficient that*:

a) *For any neighbourhood* $\mathscr{V}$ *of* H_0 *in* $\mathscr{R}^+$ *there exists an index* α_0, K *finite* $\subset I$ *and the number* $\varepsilon > 0$, $\lambda > 0$, *so that*: $\alpha \geqslant \alpha_0$ *and* $\gamma^i_E(\varepsilon) > 1 - \lambda$, $i \in K$ *imply* $\int_E f_\alpha\,\mathrm{d}\mu \in \mathscr{V}$.

b) *For any neighbourhood* $\mathscr{V}$ *of* H_0 *from* $\mathscr{R}^+$, *there exists an index* α_0 *and* $F \in \mathscr{A}$, F Γ_μ*-finite so that* $\int_E f_\alpha\,\mathrm{d}\mu \in \mathscr{V}$ *if* $\alpha \geqslant \alpha_0$ *and* $E \in \mathscr{A}$, $E \subset S - F$.

Proof. Necessity. For any neighbourhood $\mathscr{V}$ of H_0, there exists a symmetric entourage $\mathscr{W}$ of the uniform structure from $\mathscr{R}^+$ so that

$\mathscr{W}^2(H_0) \subseteq \mathscr{V}$. Let α_0 be so that $\left(\int_E f_\alpha \, d\mu, \int_E f_{\alpha_0} \, d\mu\right) \in \mathscr{W}$ for any $E \in \mathscr{A}$ if $\alpha \geqslant \alpha_0$. From the theorem 1, (iv) it results that there exists $\varepsilon > 0$, $\lambda > 0$, K finite $\subset I$, so that we have $\int_E f_{\alpha_0} \, d\mu \in \mathscr{W}(H_0)$ if $\gamma_E^i(\varepsilon) > 1 - \lambda$, $i \in K$. Therefore $\int_E f_\alpha \, d\mu \in \mathscr{V}$ if $\alpha \geqslant \alpha_0$ and $\gamma_E^i(\varepsilon) > 1 - \lambda$, $i \in K$ that, is the condition a). The condition b) is obtained by taking $F = \{s \in S;\ f_{\alpha_0}(s) \neq H_0\}$. We have $F \in \mathscr{A}$, Γ_μ-finite and $\int_E f_{\alpha_0} \, d\mu = H_0$ whichever would be $E \in \mathscr{A}$ with $E \subseteq S - F$.

Sufficiency. Let be a symmetric entourage for $\mathscr{R}_v^+$ and let α_0, K-finite $\subset I$, ε, $\lambda > 0$ and F be chosen depending on the neighbourhood $\mathscr{W}(H_0)$ according to the conditions a) and b) simultaneously.

For F and w let the entourage $\mathscr{U}$ from $\mathscr{R}^+$ be chosen according to axiom C_1. We write

$$F_{\alpha\alpha'} = \{s \in S;\ (f_\alpha(s), f_{\alpha'}(s)) \in \mathscr{U}\},\ F_{\alpha\alpha'} \in \mathscr{A}.$$

Since $\{f_\alpha\}$ is Cauchy in $[\mathscr{R}^+]^S$, there exist $\alpha_1 \geqslant \alpha_0$ so that : $\gamma^i_{F_{\alpha\alpha'}}(\varepsilon) > 1 - \lambda$, $i \in K$ for $\alpha, \alpha' \geqslant \alpha_1$. For $E \in \mathscr{A}$ in the semigroup $\mathscr{R}^+ \times \mathscr{R}^+$ we can write :

$$\left(\int_E f_\alpha \, d\mu, \int_E f_{\alpha'} \, d\mu\right) = \tau_M\left[\tau_M\left[\left(\int_{E \cap F_{\alpha\alpha'}} f_\alpha \, d\mu, \int_{E \cap F_{\alpha\alpha'}} f_{\alpha'} \, d\mu\right),\right.\right.$$

$$\left.\left(\int_{E \setminus (F_{\alpha\alpha'} \cup F)} f_\alpha \, d\mu, \int_{E \setminus (F_{\alpha\alpha'} \cup F)} f_{\alpha'} \, d\mu\right)\right], \left.\left(\int_{(E \setminus F_{\alpha\alpha'}) \cap F} f_\alpha \, d\mu, \int_{(E \setminus F_{\alpha\alpha'}) \cap F} f_{\alpha'} \, d\mu\right)\right] \in$$

$$\in \tau_M[\tau_M[(\mathscr{W}(H_0) \times \mathscr{W}(H_0)), (\mathscr{W}(H_0) \times \mathscr{W}(H_0))], \mathscr{W}] \subseteq$$

$$\subseteq \tau_M[\tau_M[\mathscr{W}^2, \mathscr{W}^2], \mathscr{W}^2] \subseteq \mathscr{W},\quad \alpha, \alpha' \geqslant \alpha_1$$

Note (i) In the previous proof, for any $\mathscr{M}, \mathscr{N} \subset \mathscr{R}^+$ we used the notation

$$\tau_M(\mathscr{M}, \mathscr{N}) = \{\tau_M(M, N);\ \forall M \in \mathscr{M},\ N \in \mathscr{N}\}$$

(ii) In the semigroup $\mathscr{R}^+ \times \mathscr{R}^+$ for $(F_1, G_1), (F_2, G_2) \in \mathscr{R}^+ \times \mathscr{R}^+$ we used the notation

$$\tau_M[(F_1, G_1), (F_2, G_2)] = (\tau_M(F_1, G_1), \tau_M(F_2, G_2))$$

LEMMA 2. *Let $\{f_\alpha\}$ and $\{g_\beta\}$ be two generalized sequences from convergent in $[\mathscr{R}^+]^S(\Gamma_\mu)$ to the same function. If $\left\{\int_E f_\alpha \, d\mu\right\}$ and $\left\{\int_E g_\beta \, d\mu\right\}$*

are generalized Cauchy sequences in $\mathscr{R}^+$ uniformly in $E \in \mathscr{A}$, then for any entourage $\mathscr{W}$ from $\mathscr{R}^+$ there exist α_0 and β_0 so that if $\alpha \geqslant \alpha_0$ and $\beta \geqslant \beta_0$, there results :

$$\left(\int_E f_\alpha \, d\mu, \int_E g_\beta \, d\mu\right) \in \mathscr{W}, \text{ uniformly in } E \in \mathscr{A}.$$

Proof. Given a symmetric entourage $\mathscr{W}$ from $\mathscr{R}^+$ so that $\tau_M[\tau_M(\mathscr{W}_1^2, \mathscr{W}_1^2], \mathscr{W}_1^2] \subseteq \mathscr{W}$ we choose an entourage $\mathscr{U}$ from $\mathscr{R}^+$ corresponding to $\mathscr{W}_1$ in conformity with axiom C_1. We write $F_{\alpha\beta} = \{s \in S, (f_\alpha(s), g_\beta(s) \notin \mathscr{U}\}$. From the previous lemma it results that there exists α_0, β_0, $\varepsilon > 0$, $\lambda > 0$, K-finite $\subset I$, so that if $F \in \mathscr{A}$ it is Γ_μ-finite and $\alpha > \alpha_0$, $\beta > \beta_0$ $\gamma_E^i(\varepsilon) > 1 - \lambda$, $i \in K$, $E \subseteq S - F$, $E \in \mathscr{A}$, imply :

$$\int_E f_\alpha \, d\mu \in \mathscr{W}_1(H_0) \text{ and } \int_E g_\beta \, d\mu \in \mathscr{W}_1(H_0).$$

By hypothesis there exist $\alpha_1 \geqslant \alpha_0$ and $\beta_1 \geqslant \beta_0$ so that for $\alpha > \alpha_1$, $\beta > \beta_1$ we have $\gamma^i_{F_{\alpha\beta}}(\varepsilon) > 1 - \lambda$, $i \in K$. Expressing the pair $\left(\int_E f_\alpha \, d\mu, \int_E g_\beta \, d\mu\right)$ the same in the proof of the sufficiency from proposition 1, the result is obtained.

DEFINITION 2. The function $f \in [\mathscr{R}^+]^S$ is called Γ_μ-integrable if there exists a generalized sequence $\{f_\alpha\}$ from $\mathscr{E}(\Gamma_\mu)$ so that $f \xrightarrow{\Gamma_\mu} f$ and $\left\{\int_E f_\alpha \, d\mu\right\}$ is a generalized Cauchy sequence in $\mathscr{R}^+$, uniformly in $E \in \mathscr{A}$.

Then the Γ_μ-integral is the element from $\mathscr{R}^+$ defined by :

$$\int_E f \, d\mu = \lim_\alpha \int_E f_\alpha \, d\mu$$

From Lemma 2, it result that the above Γ_μ-integral is properly defined. We denote by $\mathscr{L}(\Gamma_\mu)$ the set of Γ_μ-integrable functions from $[\mathscr{R}^+]^S$.

THEOREM 2.

(i) *Relatively to the operation τ_M, the set $\mathscr{L}(\Gamma_\mu)$ is a semigroup of $[\mathscr{R}^+]^S$.*

(ii) *For $E \in \mathscr{A}$ and $f, g \in \mathscr{L}(\Gamma_\mu)$ we have :*

$$\int_E \tau_M(f, g) \, d\mu = \tau_M\left(\int_E f \, d\mu, \int_E g \, d\mu\right)$$

(iii) *For* $f \in \mathscr{L}(\Gamma_\mu)$ *the mapping* $E \to \nu_E = \int_E f\,d\mu$ *is additive* :

$$\nu_{\bigcup_{i=1}^{n} E_i} = \overset{n}{\underset{i=1}{\tau_M}}(\nu_{E_i}),\ E_i \cap E_j = \Phi,\ i \neq j,\ \nu_\Phi = H_0$$

(iv) *For* $f \in \mathscr{L}(\Gamma_\mu)$ *we have* $\lim_{\substack{E \xrightarrow{\Gamma_\mu} \Phi \\ E \in \mathscr{A}}} \nu_E = H_0$

The proof follows from lemma 2 and definition 2.

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