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IMPROVED ESTIMATES WITH THE SECOND ORDER MODULUS OF CONTINUITY IN APPROXIMATION BY LINEAR POSITIVE OPERATORS

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Let I be a closed interval of the real axis, and $C(I)$ be the space of real-valued continuous functions defined on I The present paper has considered the class of linear positive operators $L : C(I) \to C(I)$, satisfying the property $L(e_j, .) = e_j$ $(j = 0, 1)$, where e_j is given by $e_j(x) = x^j$, $x \in I$, $j \geqslant 0$. This class includes some of the classical operators in the theory of approximation of continuous functions : the Bernstein operators, the Meyer-König and Zeller operators, the Szasz-Mirakjan operators, the Baskakov operators. The rate of approximation of continuous functions by the operators of this class can be expressed with the second order modulus of continuity, defined by $\omega_2(f, h) = \sup\{|f(x + \delta) - 2f(x) +$ $+ f(x - \delta)|$, $x \pm \delta \in I$, $0 < \delta \leqslant h\}$, for $f \in C(I)$, and $h > 0$. A standard technique for the study of this rate in the case where I is compact is based on the use of some functionals that admit majorants involving the second order modulus of continuity. A recent unified approach, in a more general context, that gives general and improved estimates by developing this method is due to H. H. Gonska in ([5]). In the present paper a general pointwise estimate is obtained in terms of the second modulus of continuity, based on another idea. By applying this general result to several concrete operators estimates are obtained which are better than the ones known until now.

1. General estimates. We shall use the following notation. For $a \in \mathbb{R}$ we denote by $]\,a\,[$ the greatest integer that is strictly smaller than a. If $x, y, \rho \in \mathbb{R}$, and $\rho > 0$ we put $\lambda(x, y, \rho) =]\,|x - y|/\rho\,[$. Moreover we denote by $\Phi : [0, \infty) \to \mathbb{R}$, the function defined by $\Phi(t) = \max\{7/4,$ $3/2 + t^2\}$ for $t \in [0, \infty)$. In the case where $I \subset \mathbb{R}$ is a compact interval, the space $C(I)$ is equipped with the sup-norm $\|\cdot\|$.

The theorem that we shall prove is based on the following lemmas.

LEMMA 1. *Let I be an arbitrary interval of the real axis. If $f \in C(I)$, $c \in I$, and $\delta \in \mathbb{R}$, $\delta > 0$ are such that $c \pm \delta \in I$ and $f(c - \delta) = 0 = f(c + \delta)$ then the following inequalities hold :*

(i) $\quad |f(t)| \leqslant (1/2)\,\omega_2(f, \delta)$, *if* $t = c$

(ii) $|f(t)| \leqslant \omega_2(f, \delta)$, *if* $|t - c| \leqslant \delta$

(1) (iii) $|f(t)| \leqslant (4/3)\, \omega_2(f, \delta)$, *if* $t \in I$ *and* $|t - c| \leqslant (5/3)\delta$

(iv) $|f(t)| \leqslant ((\lambda^2(c, t, \delta) + 3 \cdot \lambda(c, t, \delta))/2)\, \omega_2(f, \delta)$,

if $t \in I$ *and* $\delta < |t - c|$.

Proof. (i) We have $|f(c)| = |f(c + \delta) - 2f(c) + f(c - \delta)|/2 \leqslant (1/2)\, \omega_2(f, \delta)$.

(ii) Let $u \in [c - \delta, c + \delta]$ be such that $|f(u)| = \max\{|f(t)|, |t - c| \leqslant \delta\}$. We can assume, without any loss of generality that $u \leqslant c$. Then $2u - c + \delta \in [c - \delta, c + \delta] \subset I$ and consequently : $|f(u)| \leqslant |f(2u - c + \delta) - 2f(u) + f(c - \delta)| \leqslant \omega_2(f, \delta)$.

(iii) Let $t \in I$ be such that $|t - c| \leqslant (5/3)\delta$. We can only consider, for a choice, the case where $t \in [c - (5/3)\delta, c - \delta]$. Then $4c - 4\delta - 3t \in [c - \delta, c + \delta]$ and taking into account assertion (1) — (ii) already proved we have $|f(4c - 4\delta - 3t)| \leqslant \omega_2(f, \delta)$. Hence $|f(t)| = |(1/3) \cdot (f(4c - 4\delta - 3t) - 2f(2c - 2\delta - t) + f(t)) - (1/3)f(4c - 4\delta - 3t) + (2/3)(f(2c - 2\delta - t) - 2f(c - \delta) + f(t)| \leqslant (1/3 + 1/3 + 2/3)\, \omega_2(f, \delta)$.

(iv) Let $t \in I$ be such that $\delta < |t - c|$. We can assume that $t > c + \delta$. First let $t \in (c + \delta, c + 2\delta]$, that is $\lambda(c, t, \delta) = 1$. Hence $2c + 2\delta - t \in [c - \delta, c + \delta]$ and then by taking into account assertion (1)—(ii) we infer : $|f(t)| = |(f(t) - 2f(c + \delta) + f(2c + 2\delta - t)) + 2f(c + \delta) - f(2c + 2\delta - t)| \leqslant 2\omega_2(f, \delta)$.

Finally let $t > c + 2\delta$, and let us write $k := \lambda(c, t, \delta)$ and $y_j := c + \delta + j \cdot (t - c - \delta)/k$, $(j = 0, k)$. We have $f(t) = f(y_k) = \sum_{j=1}^{k-1} j \cdot (f(y_{k-j+1}) - 2f(y_{k-j}) + f(y_{k-j-1})) + kf(y_1) + (1 - k)\, f(y_0)$. Since $f(y_0) = f(c + \delta) = 0$ and $y_1 \in (c + \delta, c + 2\delta]$, whence from above it follows that $|f(y_1)| \leqslant 2\omega_2(f, \delta)$, we infer :

$$|f(t)| \left(\sum_{j=1}^{k-1} j + 2k\right) \omega_2(f, \delta) = ((k^2 + 3k)/2)\, \omega_2(f, \delta).$$

Thus Lemma 1 is proved.

The following lemmas will establish in certain conditions the inequality :

$$(2) \qquad |f(t) - f(x)| \leqslant \Phi(|t - x|/h)\, \omega_2(f, h), \quad t \in I,\ h > 0.$$

LEMMA 2. *Let* $I \subseteq \mathbb{R}$ *be an arbitrary interval of the real axis. Let* $f \in C(I)$, $x \in I$, *and* $h > 0$ *be such that* $x \pm h \in I$ *and* $f(x - h) = 0 = f(x + h)$. *Then inequality* (2) *holds.*

Proof. We apply Lemma 1 putting $\delta := h$ and $c := x$. If $|t - x| \leqslant h$ we have $|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant (3/2)\, \omega_2(f, h) \leqslant \Phi(|t - x|/h)\, \omega_2(f, h)$. If $kh < |t - x| \leqslant (k + 1)h$ with $k \in \mathbb{N} := \{1, 2, \ldots\}$, that is equivalently $\lambda(x, t, h) = k$, we infer $|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant ((k^2 + 3k)/2) + 1/2\omega_2(f, h)$. On the other hand we obtain $\Phi(|t - x|/h) \geqslant 3/2 + (t - x)^2/h^2 \geqslant 3/2 + k^2$. Since the inequality $k^2 + 3k + 1 \leqslant 2k^2 + 3$ holds for any $k \in \mathbb{N}$, Lemma 2 is proved.

LEMMA 3. *Let* $I \subset \mathbb{R}$ *be a closed below bounded interval of the real axis, and let us denote by* $a \in \mathbb{R}$, $a = \min I$, *and by* $b \in \mathbb{R} \cup \{\infty\}$ $b = \sup I$. *Let* $x \in I$ *and* $h > 0$ *be such that* $x < a + h$. *In the case where* $b \in \mathbb{R}$ *we assume the additional condition* $x \leqslant (3a + b)/4$. *Then if* $f \in C(I)$ *satisfies* $f(x) = 0 = f(d)$, *where* d *denotes* $d = \min\{b, x + 2h\}$, *inequality* (2) *holds.*

Proof. If $d = b$ we shall use Lemma 1 by taking $\delta = (b - x)/2$ and $c = (b + x)/2$. Thus for $t \in [x, b]$, since $\delta \leqslant h$ we have $|f(t) - f(x)| = |f(t)| \leqslant \omega_2(f, \delta) \leqslant \Phi(|t - x|/h)\, \omega_2(f, h)$ and if $t \in [a, x)$ since $|t - c| = c - t \leqslant c - a = \delta + x - a \leqslant \delta + (b - a)/3 - (x - a)/3 = (5/3)\, \delta$, and $\delta \leqslant h$ we have $|f(t) - f(x)| = |f(t)| \leqslant (4/3)\, \omega_2(f, \delta) \leqslant \Phi(|t - x|/h)\, \omega_2(f, h)$.

Now let us assume that $d = x + 2h$. We shall consider the intervals $I_k = [x - h + h/3^k,\ x + 2h]$ and $J_k = [x,\ x + 3h - h/3^k]$, for $k = 0, 1, 2, \ldots$ Therefore we have $I_k \subset I_{k+1}$, $J_k \subset J_{k+1}$ for $k = 0, 1, 2 \ldots$ and $I_0 = J_0 = [x, x + 2h]$. We denote by S the interval $S = (a, \max\{x + 2h, 4x - 3a\})$. If $b \in \mathbb{R}$ we have $4x - 3a \leqslant b$. Hence $S \subset I$ for every $b \in \mathbb{R} \cup \{\infty\}$. We shall prove by induction the inequality :

$$(*) \qquad |f(t)| \leqslant ((3 - 1/3^k)/2)\, \omega_2(f, h), \text{ for } t \in (I_k \cup J_k) \cap S,$$

and $k = 0, 1, 2, \ldots$

Indeed for $k = 0$ putting $c = x + h$ and $\delta := h$, the inequality $(*)$ is a simple consequence of Lemma 1 — (ii). Let now $k > 0$ and suppose relation $(*)$ true for $k - 1$. We have to consider two nondisjoint cases $t \in I_k \cap S$ and $t \in J_k \cap S$.

In the first case let $t \in I_k \cap S$. If $t \geqslant x$ then $t \in I_0$ and relation $(*)$ is immediate. If $t < x$ then $4x - 3t \geqslant x$ and on the other hand the inequality $t \geqslant x - h + h/3^k$ implies $4x - 3t \leqslant x + 3h - h/3^{k-1}$ and then $4x - 3t \in J_{k-1}$. Moreover if $b \in \mathbb{R}$ we have $4x - 3t < 4x - 3a \leqslant b$. Hence $4x - 3t \in J_{k-1} \cap S$. Using the assumption of induction we have

$$|f(t)| = |(1/3)(f(4x - 3t) - 2f(2x - t) + f(t)) - (1/3)f(4x - 3t) + (2/3)(f(2x - t) - 2f(x) + f(t))| \leqslant$$
$$\leqslant (1/3 + (1/3)(3 - 1/3^{k-1})/2 + 2/3)\, \omega_2(f, h) = ((3 - 1/3^k)/2)\, \omega_2(f, h).$$

In the second case let $t \in J_k \cap S$. If $t \leqslant x + 2h$ then $t \in J_0$, and relation $(*)$ is immediate. If $t > x + 2h$ it follows that $4x + 8h - 3t < x + 2h$, and on the other hand the inequality $t \leqslant x + 3h - h/3^k$ implies $4x + 8h - 3t \geqslant x - h + h/3^{k-1}$. Hence $4x + 8h - 3t \in I_{k-1} \cap S$. Using the assumption of induction we have :

$$|f(t)| = |(1/3)(f(4x + 8h - 3t) - 2f(2x + 4h - t) + f(\bar{t})) + (2/3)(f(2a + 4h - t) - 2f(x + 2h) + f(t)) - (1/3)f(4x + 8n - 3t)| \leqslant$$
$$\leqslant (1/3 + 2/3 + (1/3)(3 - 1/3^{k-1})/2)\, \omega_2(f, h) = ((3 - 1/3^k)/2)\, \omega_2(f, h).$$

The above inequalities prove $(*)$.

From $(*)$ it follows that for $t \in (a, x + 2h]$ we have $|f(t)| \leqslant (3/2)\omega_2(f, h)$, and then using the continuity of the function $|f|$, this inequality can be extended in the point $a \in I$ too. Hence, finally we have:

$$(**) \qquad |f(t)| \leqslant (3/2)\,\omega_2(f, h) \text{ for } t \in I, \text{ and } |t - x| \leqslant 2h.$$

If $t \in I$ and $|t - x| > 2h$ it follows that $t > x + 2h$ and taking into account Lemma 1 — (iv) and the fact that $\lambda(x + h, t, h) = \lambda(x, t, h) - 1$ we obtain: $|f(t)| \leqslant ((\lambda^2(x+h, t, h) + 3\lambda(x+h, t, h))/2)\,\omega_2(f, h) = ((\lambda^2(x, t, h) + \lambda(x, t, h) - 2)/2)\,\omega_2(f, h)$.

If we write $k = \lambda(x, t, h)$, from the inequality $k^2 + k - 2 \leqslant 2k^2 + 3$, that is true for any $k \geqslant 1$ and from the inequality $k \cdot h < |t - x|$ it follows:

$$(***) \qquad |f(t)| \leqslant (3/2 + (t - x)^2/h^2)\,\omega_2(f, h), \text{ for } t \in I, \text{ and } |t - x| > 2h.$$

The inequalities in $(**)$ and $(***)$ show the validity of the Lemma.

LEMMA 4. *Let* $I = [a, b] \subset \mathbb{R}$, $x \in ((3a + b)/4, (a + b)/2]$ *and* $h > 0$ *be such that* $x < a + h$. *If* $f \in C(I)$ *satisfies the assumption*: $f(a) = 0 = f(d)$, *where* $d = \min\{b, a + 2h\}$ *then inequality* (2) *holds.*

Proof. Taking $c := (a + d)/2$ and $\delta = (d - a)/2$ in Lemma 1 — (ii), it follows that $|f(t)| \leqslant \omega_2(f, h)$ for $t \in [a, d]$. By the assumption in Lemma 4 we have $[a, 2x - a] \subset [a, d] \subset I$. First we shall prove the inequality:

$$(*) \qquad |f(t) - f(x)| \leqslant (3/2)\,\omega_2(f, h) \text{ for } t \in [a, 2x - a].$$

Indeed let $t \in [a, x)$. Then we have $2x - t \in (x, 2x - a]$, and hence

$$|f(t) - f(x)| = |(1/2)\,(f(2x - t) - 2\cdot f(x) + f(t)) + (1/2)\,f(t) - (1/2)\cdot f(2x - t)| \leqslant (3/2)\,\omega_2(f, h).$$

The case $t \in (x, 2x - a]$ is analogous.

Now we shall prove the inequality:

$$(**) \qquad |f(t) - f(x)| \leqslant (7/4)\,\omega_2(f, h) \text{ for } t \in [a, d].$$

Indeed, let $t \in (2x - a, d]$. From the assumption in Lemma 4 it follows that $4x - 3a \geqslant d$ and consequently $4x - 2a - t \in [a, 2x - a] \subset [a, d]$. We have

$$|f(t) - f(x)| = |(1/4)(f(4x - 2a - t) - 2f(2x - a) + f(t)) + (3/4)\,f(t) + (1/2)(f(2x - a) - 2f(x) + f(a)) - (1/4)\,f(4x - 2a - t)| \leqslant$$

$$\leqslant (1/4 + 3/4 + 1/2 + 1/4)\,\omega_2(f, h) = (7/4)\,\omega_2(f, h).$$

Finally we shall prove the inequality:

$$(***) \qquad |f(t) - f(x)| \leqslant (3/2 + (t - x)^2/h^2)\omega_2(f, h),\ t \in I \cap (d, \infty).$$

Indeed in this case $d = a + 2h$. In what follows we shall apply Lemma 1 — (iii) and (iv), by choosing $c := a + h$ and $\delta := h$.

If $t \in I \cap (a + 2h, a + (8/3)h]$ we have $|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant (4/3)\omega_2(f, h) + \omega_2(f, h) = (7/3)\omega_2(f, h)$ and on the other hand we obtain $3/2 + (t - x)^2/h^2 \geqslant 3/2 + 1 > 7/3$.

If $t \in I \cap (a + (8/3)h, a + 3h]$ we have $|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant 3\omega_2(f, h)$, and on the other hand $3/2 + (t - x)^2/h^2 \geqslant 3/2 + (5/3)^2 > 3$.

If $t \in (a + 3h, a + (10/3)h] \cap I$ we have $(t + a + 2h)/2 \in (a + 2h, a + (8/3)h]$. Therefore we have $|f(t) - f(x)| \leqslant |f(t)| + |f(x)| = |(f(t) - 2\cdot f((t + a + 2h)/2) + f(a + 2h)) + 2\cdot f((a + t + 2h)/2))| + |f(x)| \leqslant (1 + 8/3 + 1)\,\omega_2(f, h) = (14/3)\,\omega_2(f, h)$. On the other hand $3/2 + (t - x)^2/h^2 \geqslant 3/2 + 4 = 11/2 > 14/3$.

If $t \in (a + (10/3)h, a + 4h] \cap I$ then $\lambda(a + h, t, h) = 2$ and hence

$$|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant$$

$$\leqslant ((\lambda^2(a + n, t, h) + 3\lambda(a + h, t, h) + 2)/2)\,\omega_2(f, h) = 6\omega_2(f, h),$$

and on the other hand one obtains $3/2 + (t - x)^2/h^2 > 3/2 + (7/3)^2 = 125/18 > 6$.

If $t \in (a + 4h, \infty) \cap I$ by denoting $k = \lambda(x, t, h)$, we have

$$|f(t) - f(x)| \leqslant |f(t)| + |f(x)| \leqslant$$

$$\leqslant ((\lambda^2(a + h, t, h) + 3\lambda(a + h, t, h) + 2)/2)\omega_2(f, h) \leqslant$$

$$\leqslant ((k^2 + 3k + 2)/2)\,\omega_2(f, h).$$

On the other hand it obtains $3/2 + (t - x)^2/h^2 \geqslant 3/2 + k^2$. Since the inequality $3/2 + k^2 > (k^2 + 3k + 2)/2$ holds for any $k \geqslant 3$, the inequality in $(***)$ is completely proved. The Lemma is a consequence of relations $(**)$ and $(***)$.

The following lemma summarizes Lemmas 2—4.

LEMMA 5. *Let* I *be an arbitrary interval of the real axis. Then for every function* $f \in C(I)$ *and any point* $x \in I$ *there exists a polynomial* p_x *of degree* I *such that the function* g_x *defined by*: $g_x := f + p_x$ *to verify the inequality*:

$$(2') \qquad |g_x(t) - g_x(x)| \leqslant \Phi(|t - x|/h)\,\omega_2'(g_x, h), \quad t \in I,\ h > 0.$$

Proof. In order to find such a polynomial, let us denote for $\alpha \in I$, $\beta \in I$, $\alpha < \beta$ by $q_{\alpha,\beta}$ the polynomial of degree I defined by $q_{\alpha,\beta}(t) = -f(\alpha) - ((f(\beta) - f(\alpha))/(\beta - \alpha))\,(t - \alpha)$, $t \in I$. Then the function $g = f + q_{\alpha,\beta}$ satisfies $g(\alpha) = 0 = g(\beta)$. We have to consider several cases.

Case 1: $I = \mathbb{R}$. In this case $[x - h, x + h] \subset I$, we take $p_x = q_{x-h,x+h}$ and we can apply Lemma 2.

Case 2: I is of the form $I = [a, \infty)$, $a \in \mathbb{R}$.

(i) If $x \geqslant a + h$ we have $[x - h, x + h] \subset I$, and we can apply Lemma 2 taking $p_x := q_{x-h,x+h}$.

ii) If $a \leqslant x < a+h$ we put $p_x := q_{x,x+2h}$ and we can apply Lemma 3.

Case 2': I is of the form $I = (-\infty, b]$, $b \in \mathbb{R}$. This case can be reduced to Case 2. Indeed, let us denote $I_1 = \{t, \ -t \in I\}$ and $s : I \to I_1$, $s(t) = -t$, $(t \in I)$. Then $-x \in I_1$ and for the function $f \circ (s)^{-1} \in C(I_1)$ there exists a polynomial p^*_{-x} of degree I such that for the function $g^*_{-x} = f \circ (s)^{-1} + p^*_{-x}$ to satisfy the inequality:

$$|g^*_{-x}(t) - g^*_{-x}(-x)| \leqslant \Phi(|t + x|/h)\omega_2(g^*_{-x}, h), \ t \in I_1, \ h > 0.$$

The inequality in (2') follows from the above inequality by taking $p_x = p^*_{-x} \circ s$.

Case 3: I is of the form $I = [a, b]$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a \leqslant x \leqslant (a + b)/2$.

(i) If $x \geqslant a + h$ then $[x - h, x + h] \subset I$, we take $p_x = q_{x-h,x+h}$ and we can apply Lemma 2.

(ii) If $a \leqslant x \leqslant a + h$ and $x \leqslant (3a + b)/4$ we take $p_x = q_{x,d}$, where $d = \min\{b, x + 2h\}$ and we can apply Lemma 3.

(iii) If $a \leqslant x \leqslant a + h$ and $x > (3a + b)/4$ we take $p_x = q_{a,d}$, where $d = \min\{b, a + 2h\}$ and we can apply Lemma 4.

Case 3': I is of the form $I = [a, b]$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $(a + b)/2 \leqslant x \leqslant b$. This case can be reduced to the Case 3 similarly as the reduction of the Case 2' to the Case 2.

Thus assertion (2') is proved in all the cases.

Now we shall give the main result:

THEOREM 1. *If $I \subset \mathbb{R}$ is an arbitrary closed interval of the real axis and $L : C(I) \to C(I)$ is a linear positive operator that satisfies the property $L(e_j, x) = x^j$, $j = 0,1$, then the following estimate:*

$$|L(f, x) - f(x)| \leqslant \max\{7/4, \ 3/2 + L((e_1 - x)^2, x)/h^2\} \ \omega_2(f, h) \tag{3}$$

holds for any $f \in C(I)$, $h > 0$, and $x \in I$.

Proof. Let $f \in C(I)$, $h > 0$ and $x \in I$ fixed. From Lemma 5 there exists a polynomial p_x of degree I such that the function $g_x = f + p_x$ satisfies relation (2'). Taking into account the property of the operator L and the equality $\omega_2(g_x, h) = \omega_2(f, h)$ we have:

$$|L(f, x) - f(x)| = |L(g_x, x) - g_x(x)| = |L(g_x - g_x(x), x)| \leqslant$$
$$\leqslant L(|g_x - g_x(x)|, x) \leqslant L(\Phi(|e_1 - x|/h), x)\omega_2(g_x, h) \leqslant$$
$$\leqslant \max\{7/4, \ 3/2 + L((e_1 - x)^2, x)/h^2\}\omega_2(f, h).$$

COROLLARY 1. ***Let $I \subset \mathbb{R}$ be a closed interval of the real axis and $L : C(I) \to C(I)$ be a linear positive operator that satisfies the property $L(e_j, x) = x^j$, $j = 0, 1$. Then for any $f \in C(I)$, any number $k > 0$ and any point $x \in I$ the following estimates hold:***

$$|L(f, x) - f(x)| \leqslant \max\{7/4, \ 3/2 + k\} \ \omega_2(f, (L((e_1 - x)^2, x)/k)^{1/2}) \tag{4}$$

and for $k = 1$ it follows:

$$|L(f, x) - f(x)| \leqslant (5/2)\omega_2(f, (L((e_1 - x)^2, x)^{1/2}). \tag{5}$$

COROLLARY 2. *Let $I \subset R$ be a compact interval of the real axis and $L : C(I) \to C(I)$ be a linear positive operator that satisfies the property $L(e_j, x) = x^j$, $j = 0, 1$. Then for any $f \in C(I)$ and $h > 0$ the following estimate holds:*

$$\|L(f, \cdot) - f\| \leqslant \max\{7/4, \ 3/2 + \sigma(L)/h^2\}\omega_2(f, h), \tag{6}$$

where $\sigma(L) := \max\{L((e_1 - x)^2, x), \ x \in I\}$.

2. Applications to several linear positive operators. We shall now apply Theorem 1 and its Corollaries in order to establish estimates for certain well-known linear positive operators.

A. The Bernstein operators. Let $B_n : C([0, 1]) \to C([0, 1])$ be the Bernstein operators given by formula:

$$B_n(f, x) = \sum_{j=0}^{n} f(j/n) \ p_{nj}(x), \text{ where } p_{nj}(x) := \binom{n}{j} \cdot x^j(1 - x)^{n-j},$$

for $x \in [0, 1]$, $n = 1, 2, \ldots$

THEOREM 2a. *We have:*

$$\|B_n(f, \cdot) - f\| \leqslant (1{,}75)\omega_2(f, n^{-1/2}), \tag{7}$$

for any $f \in C([0, 1])$ and $n = 1, 2, \ldots$

Proof. $h := n^{-1/2}$. Then $\sigma(B_n)/h^2 = \max\{x(1 - x), \ x \in [0, 1]\} = 1/4$, and we find $3/2 + \sigma(B_n)/h^2 = 7/4$. Then we apply Corollary 2.

Remark. The value equal to 1.75 of the constant in Theorem 2a refines the best known value equal to 3.25 given first by Brudnyi ([2]).

A pointwise version of Theorem 2a is the following:

THEOREM 2b. *For any $f \in C([0, 1])$, $x \in [0, 1]$, and $n = 1, 2, \ldots$ there holds:*

$$|B_n(f, x) - f(x)| \leqslant (1.75)\omega_2\left(f, \left(\frac{4x(1 - x)}{n}\right)^{1/2}\right). \tag{8}$$

Proof. It follows from Corollary 1 — (4) by taking $k = 1/4$.

Remark. The value equal to 1.75 of the constant in assertion (8) improves the value equal to 3.25 given in the corresponding pointwise version estimate given by H. Gonska ([5]).

THEOREM 2c. *For any $f \in C([0, 1])$, $x \in [0, 1]$ and $n = 1, 2, \ldots$ there holds:*

$$|B_n(, x) - f(x)| \leqslant (2{,}5)\omega_2\left(f, \left(\frac{x(1 - x)}{n}\right)^{1/2}\right). \tag{9}$$

Proof. It follows from Corollary 1 — (5).

Remark The value equal to 2.5 of the constant in inequality (9) improves the value equal to 4 given in the estimate in ([5]).

B. The Meyer-König and Zeller operators. Let $M_n : C([0, 1]) \to C([0, 1])$ be the operators of Meyer-König and Zeller defined by

$$M_n(f, x) = (1-x)^{n+1}) \sum_{k=0}^{\infty} \binom{n+k}{k} \cdot x^k \cdot f(k/(n+k)), \text{ for } 0 \leqslant x < 1, \text{ and}$$

$$M_n(f, 1) = f(1)., \text{ where } n = 1, 2, \ldots$$

THEOREM 3. *For any* $f \in C([0, 1])$ *and* $n = 1, 2, \ldots$ *there holds*:

(10) $$\|M_n(f, \cdot) - f\| \leqslant (1.75)\omega_2(f, (n+1)^{-1/2}).$$

Proof. With the help of the inequality given by J. Alkemade ([1]): $\sigma(M_n) \leqslant 4/(27n + 9)$, by taking $h = (n+1)^{-1/2}$ it follows $\sigma(M_n)/h^2 \leqslant 2/9$. We give $3/2 + \sigma(M_n)/h^2 \leqslant 31/18 < 7/4$, and we can apply Corollary 2.

Remark. The value equal to 1.75 of the constant in Theorem 3 improves the value equal to $3\frac{4}{21}$ in the estimate given in ([5]).

C. The Szasz-Mirakjan operators. The Szasz-Mirakjan operators $S_n : C([0, \infty)) \to C([0, \infty))$ are given by the formula:

$$S_n(f, x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \cdot f(k/n), \quad x \in [0, \infty), \; n = 1, 2, \ldots$$

We have $S_n((e_1 - x)^2, x) = x/n$ and from Theorem 1 and respectively from Corollary 1 – (5) one obtains:

THEOREM 4. *For any* $f \in C([0, \infty))$, $x \in [0, \infty)$ *and* $n = 1, 2, \ldots$ *the following estimates hold*:

(11) $$|S_n(f, x) - f(x)| \leqslant \max\{7/4, 3/2 + x\}\omega_2(f, n^{-1/2})$$

(12) $$|S_n(f, x) - f(x)| \leqslant (2.5)\,\omega_2(f, (x/n)^{1/2}).$$

D. The Baskakov operators. The Baskakov operators $V_n : C([0, \infty)) \to C([0, \infty))$ are defined by:

$$V_n(f, x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k} f(k/n), \quad x \in [0, \infty),$$

$$n = 1, 2, \ldots$$

We have $V_n((e_1 - x)^2, x) = \dfrac{x(1+x)}{n}$. From assertion (3) and (5) there results

THEOREM 5. *For any* $f \in C([0, \infty))$, $x \in [0, \infty)$ *and* $n = 1, 2, \ldots$ *the following estimates hold*:

(13) $$|V_n(f, x) - f(x)| \leqslant \max\{7/4, 3/2 + x(1+x)\}\omega_2(f, n^{-1/2}).$$

(14) $$|V_n(f, x) - f(x)| \leqslant (2.5)\omega_2\left(f, \left(\frac{x(1+x)}{n}\right)^{1/2}\right).$$

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