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ON SOME SYSTEMS OF LINEAR INEQUALITIES

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1. Let X be a linear space over $\mathbb{R}$ and let $X^{\#}$ be the algebraic dual of X. Let S be a nonempty set and $A = \{a_s : s \in S\} \subset X^{\#}$. Let $b \in X^{\#}$. Consider the following systems of linear inequalities:

$$S_1 \begin{cases} a(x) \geqslant 0 & \text{for all} \quad a \in A \\ b(x) < 0 \end{cases}$$

$$S_2 \begin{cases} a(x) > 0 & \text{for all} \quad a \in A \\ b(x) \leqslant 0 \end{cases}$$

We shall study the relationship between the inconsistency of S_1 and that of S_2 and we shall give some applications.

2. Throughout the paper we shall suppose that:

(I) There exists $x_0 \in X$ such that $b(x_0) = 1$ and $a(x_0) = 1$ for all $a \in A$.

THEOREM 1. *Let us consider the following statements:*

(1) *For each* $x \in X$, $b(x) \leqslant \sup\{a(x) : a \in A\}$

(2) S_1 *is inconsistent.*

(3) S_2 *is inconsistent.*

(4) *For each* $x \in X$ *there is* $a \in A$ *such that* $b(x) \leqslant a(x)$

(5) *For each* $x \in X$ *there is* $x^{\#} \in \operatorname{conv}(A)$ *such that* $b(x) = x^{\#}(x)$.

Then (1) *and* (2) *are equivalent,* (3), (4) *and* (5) *are equivalent and* (4) *implies* (1).

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ be a solution of S_1. Then $b(x) < 0$ and $b(-x) \leqslant \sup\{a(-x) : a \in A\} = -\inf\{a(x) : a \in A\} \leqslant 0$, hence $b(x) \geqslant 0$, a contradiction.

(2) $\Rightarrow$ (1). Let $x \in X$ and $M = \sup\{a(x) : a \in A\} < \infty$. Let $y = Mx_0 - x$. Then $a(y) \geqslant 0$ for all $a \in A$; since S_1 is inconsistent, we obtain $b(y) \geqslant 0$, hence $b(x) \leqslant M$.

(3) $\Rightarrow$ (4). Let $x \in X$, $y = b(x)x_0 - x$. Then $b(y) = 0$; since S_2 is inconsistent, there exists $a \in A$ such that $a(y) \leqslant 0$. It follows that $b(x) \leqslant a(x)$.

(4) $\Rightarrow$ (5). Let $x \in X$. Using (4) we deduce that there are $a', a'' \in A$ such that $a'(x) \leqslant b(x) \leqslant a''(x)$. If $a'(x) = a''(x)$, put $x^{\#} = a'$; if $a'(x) <$

$< a''(x)$, put $x^{\#} = [(a''(x) - b(x))a' + (b(x) - a'(x))a'']/(a''(x) - a'(x))$. Then $x^{\#} \in \mathrm{conv}(A)$ and $x^{\#}(x) = b(x)$.

(5) $\Rightarrow$ (3). Let $x \in X$ be a solution of S_2. Then $b(x) \leqslant 0$. Let $x^{\#} \in \mathrm{conv}\,(A)$, $b(x) = x^{\#}(x)$. It follows that $b(x) > 0$, a contradiction.

Since obviously (4) implies (1), the proof is complete.

EXAMPLE 1. Let $X = C^2[-1, 1]$, $A = \{[t_1, t_2, t_3\,;\ \cdot]: -1 \leqslant t_1 < t_2 < t_3 \leqslant 1\}$, $b(x) = (1/2)\, x''(0)$ for all $x \in X$. Then (I) is satisfied ($x_0(t) = t^2$). S_1 is inconsistent, but $x(t) = t^4$ is a solution of S_2.

Hence (2) *does not imply* (3).

3. For $x \in X$ let us denote by v_x the function

$$v_x : S \to R, \quad v_x(s) = a_s(x).$$

THEOREM 2. *Let S be a connected topological space and suppose that for each $x \in X$ the function v_x is continuous on S. Then* (3) *is equivalent to*:

(6) *For each $x \in X$ there exists $a \in A$ such that $b(x) = a(x)$.*

Proof. Since (6) $\Rightarrow$ (5) $\Rightarrow$ (3), it remains to prove that (3) implies (6).

Let $x \in X$. Since (3) implies (5), there exists $x^{\#} \in \mathrm{conv}(A)$ such that $b(x) = x^{\#}(x)$. Hence $b(x) \in \mathrm{conv}\,\{a_s(x) : s \in S\}$. The function $s \to a_s(x)$ is continuous on the connected space S; it follows that $\{a_s(x) : s \in S\}$ is an interval. Therefore $b(x) \in \{a_s(x) : s \in S\}$ and the proof is finished.

EXAMPLE 2. Let $0 \leqslant k \leqslant n$ and let $X = C^k[\alpha, \beta]$ be endowed with the norm

$$\|f\|_k = \max\,\{\|f\|, \|f'\|, \ldots, \|f^{(k)}\|\},$$

$\|\cdot\|$ being the sup-norm. Let X' be the topological dual of X.

Let $S = \{(t_1, \ldots, t_{n+2}) \in \mathbb{R}^{n+2} : \alpha \leqslant t_1 < \ldots < t_{n+2} \leqslant \beta\}$. For $s = (t_1, \ldots, t_{n+2}) \in S$, let $a_s = [t_1, \ldots, t_{n+2}\,;\ \cdot]$.

Let $x_0(t) = t^{n+1}$ and let $b \in X'$ be such that $b(x_0) = 1$. Then (I) is satisfied.

Using Theorem 1 we deduce that if S_2 is inconsistent, then S_1 is inconsistent. The converse is also true; see [2], Theorem 1.

COROLLARY 1 (T. Popoviciu). *If S_1 is inconsistent, then for each $f \in C^k[\alpha, \beta]$ there exists $s \in S$ such that $b(f) = a_s(f)$.*

Proof. Let S_1 be inconsistent. By the above remark, S_2 is also inconsistent. S is a connected subspace of $\mathbb{R}^{n+2}$ and the function $s \to a_s(f)$ is continuous for each $f \in C^k[\alpha, \beta]$. Now it suffices to apply Theorem 2.

EXAMPLE 3. Let K be a metrizable compact convex subset of a locally convex Hausdorff space over $\mathbb{R}$. Let $X = C(K)$ be endowed with the sup-norm and let

$$S = \{(t_1, t_2, c) : t_1, t_2 \in K,\ t_1 \neq t_2,\ c \in (0, 1)\}.$$

Let $x_0 \in C(K)$ be a strictly convex function.

For $s = (t_1, t_2, c) \in S$ and $x \in C(K)$ let

$$a_s(x) = [(1 - c)\, x(t_1) + cx(t_2) - x((1 - c)t_1 + ct_2)]/ /[(1 - c)\, x_0(t_1) + cx_0(t_2) - x_0((1 - c)t_1 + ct_2)].$$

Let $b \in X'$ be such that $b(x_0) = 1$.

By Theorem 1, if S_2 is inconsistent, then S_1 is inconsistent. The converse is also true; see [3], Corollary 1.

COROLLARY 2. ([4], Th.2). *If S_1 is inconsistent, then for each $x \in C(K)$ there exists $s \in S$ such that $b(x) = a_s(x)$.*

Proof. Let S_1 be inconsistent. By the above remark, S_2 is also inconsistent.

Let $\Delta = \{(t, t) : t \in K\}$. The product $K \times K$ is connected and $(K \times K) \setminus \Delta = C_1 \cup C_2$, where C_1 and C_2 are connected components; we have $(t_1, t_2) \in C_1$ iff $(t_2, t_1) \in C_2$. Moreover,

$$S = [(K \times K) \setminus \Delta] \times (0, 1) = [C_1 \times (0, 1)] \cup [C_2 \times (0, 1)].$$

Let $S' = C_1 \times (0, 1)$.

It is easy to see that

$$a_{(t_1, t_2, c)} = a_{(t_2, t_1, 1-c)}$$

Since S_2 is inconsistent it follows that the following system:

$$\begin{cases} a_{s'}(x) > 0 & \text{for all } s' \in S' \\ b(x) \leqslant 0 \end{cases}$$

is also inconsistent. But S' is connected; an application of Theorem 2 finishes the proof.

REMARK. Applications of Corollaries 1 and 2 are given in [2], [4].

4. Let $B(S)$ be the space of all real-valued, bounded functions on S.

THEOREM 3. *Suppose that for each $x \in X$, $v_x \in B(S)$. Then* (2) *is equivalent to*:

(7) *There exists a positive linear functional p on $B(S)$ such that $b(x) = p(v_x)$ for all $x \in X$.*

Proof. (2) $\Rightarrow$ (7). Since (2) implies (1), we can use Erweiterter Maximumssatz of H. König [1]; it follows that there exists $p \in B(S)^{\#}$ such that

(i) $p(v) \leqslant \sup\{v(s) : s \in S\}$ for all $v \in B(S)$, and

(ii) $b(x) \leqslant p(v_x)$ for all $x \in X$.

Let $v \in B(S)$, $v \leqslant 0$. Using (i) we obtain $p(v) \leqslant 0$; therefore p is a positive linear functional.

Let $x \in X$. From (ii) it follows that $b(-x) \leqslant p(v_{-x}) = p(-v_x)$, i.e., $b(x) = p(v_x)$.

(7) $\Rightarrow$ (2). Let $x \in X$ be a solution of S_1. Then $a_s(x) \geqslant 0$ for all $s \in S$, hence $v_x \geqslant 0$. It follows that $b(x) = p(v_x) \geqslant 0$. But $b(x) < 0$, a contradiction.

THEOREM 4. *If S is a compact Hausdorff space and $v_x \in C(S)$ for all $x \in X$, then* (2), (3) *and*
(8) *there exists a probability Radon measure p on S such that $b(x) = \int_S v_x(s)dp(s)$ for all $x \in X$ are equivalent.*

Proof. (3) $\Rightarrow$ (2). See Theorem 1.
(2) $\Rightarrow$ (8). The positive linear functional p on $B(S)$ given by Theorem 3 satisfies $p(1) = p(v_{x_0}) = b(x_0) = 1$. Hence the restriction of p to $C(S)$ can be identified with a probability Radon measure on S.
(8) $\Rightarrow$ (3). Let $x \in X$ be a solution of S_2. Then $b(x) \leqslant 0$ and $a_s(x) > 0$, hence $v_x(s) > 0$ for all $s \in S$. It follows that $m : = \min \{v_x(s) : s \in S\} > 0$. Then $b(x) = \int_S v_x(s)dp(s) \geqslant m > 0$, which contradicts $b(x) \leqslant 0$.

EXAMPLE 4. (see also [1]). Let $a_n \in X^{\#}$ be such that
(i) For all $x \in X$ there exists $a_\infty(x) = \lim a_n(x) \in \mathbb{R}$, and
(i) There exists $x_0 \in X$ with $a_n(x_0) = 1$, $n = 1, 2, \ldots$.
Then $a_\infty \in X^{\#}$ Let $S = \mathbb{N} \cup \{\infty\}$ be the Alexandrov one-point compactification of the discrete space $\mathbb{N}$. It is easy to verify that $v_x \in C(S)$ for all $x \in X$.
Let $b \in X^{\#}$ be such that $b(x_0) = 1$. From Theorem 1 and Theorem 4 it follows that (1)–(5) and (8) are equivalent. This means that the following statements are equivalent:
(1′) $b(x) \leqslant \sup\{a_n(x) : n \in N\}$ for all $x \in X$
(2′) $b(x) \geqslant 0$ for all $x \in X$ with $a_n(x) \geqslant 0$, $n = 1,2, \ldots$
(3′) $b(x) > 0$ for all $x \in X$ with $a_n(x) > 0$, $n = 1,2, \ldots$ and $\lim_{n\to\infty} a_n(x) > 0$
(4′) for each $x \in X$ there exists $s \in N \cup \{\infty\}$ such that $b(x) \leqslant a_s(x)$.
(5′) for each $x \in X$ there exists $x^{\#} \in \mathrm{conv}\{a_s : s \in \mathbb{N} \cup \{\infty\}\}$ such that $b(x) = x^{\#}(x)$.
(8′) there are $c_\infty, c_1, c_2, \ldots \geqslant 0$, $c_\infty + c_1 + c_2 + \ldots = 1$, such that $b(x) = c_\infty \lim_{n\to\infty} a_n(x) + c_1a_1(x) + c_2a_2(x) + \ldots$ for all $x \in X$.

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