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ON THE MONOTONICITY OF SEQUENCES OF BERNSTEIN-SCHNABL OPERATORS

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Let X be a compact convex subset of a locally convex Hausdorff space over R. For each $x \in X$, let μ_x be a probability Radon measure on X having x as barycenter. Let $\mu_{x,n} = \mu_x \otimes \ldots \otimes \mu_x$, the number of factors being n.

For $x \in X$, $n \geqslant 1$ and $f \in C(X)$ let

$$B_n f(x) = \int_{X^n} f\left(\frac{t_1 + \ldots + t_n}{n}\right) \mathrm{d}\mu_{x,n}(t_1, \ldots, t_n)$$

B_n are Bernstein-Schnabl type operators (see [2]). The classical operators of Bernstein, the operators B_n^{Δ} from [13] $B_{n,m}$, $B^*_{n,m}$ from [12] $B_{n,m}$ and B_n from [6, Sect. 3] can be obtained in this way.

As a consequence of Theorem 2 in [2] we have

(1) $\lim\limits_{n\to\infty} B_n f(x) = f(x)$ *uniformly on* X, *for all* $f \in C(X)$.

Let $f \in C(X)$ be a *convex* function, $n \geqslant 1$ and $x \in X$; then

(2) $B_n f(x) \geqslant f(x)$

(3) $B_n f(x) = f(x)$ *iff* f *is affine on* $\mathrm{cl}(\mathrm{conv}(\mathrm{supp}\ \mu_x))$.

(See [9], (16) and (17)).

The aim of this note is to prove the following

THEOREM. *If* $f \in C(X)$ *is a convex function, then*

(4) $B_n f(x) \geqslant B_{n+1} f(x)$ *for all* $x \in X$ *and all* $n \geqslant 1$.

For other types of generalized Bernstein operators such inequalities have been studied in [1], [4], [6], [10], [11], [13]. Converse results have been obtained in [3], [5], [7], [14].

Proof of the Theorem. For $(t_1, \ldots, t_{n+1}) \in X^{n+1}$ and $i \in \{1, \ldots, n+1\}$ let us write

$$\bar{t}_i = \frac{1}{n}\left(\sum_{j=1}^{n+1} t_j - t_i\right).$$

Then we have

$$B_n f(x) - B_{n+1} f(x) = \int_{X^{n+1}} f\left(\frac{t_1 + \ldots + t_n}{n}\right) d\mu_{x,n+1}(t_1, \ldots, t_{n+1}) -$$

$$- \int_{X^{n+1}} f\left(\frac{t_1 + \ldots + t_{n+1}}{n+1}\right) d\mu_{x,n+1}(t_1, \ldots, t_{n+1}) =$$

$$= \int_{X^{n+1}} \frac{1}{n+1} \sum_{i=1}^{n+1} f(\bar{t}_i)\, d\mu_{x,n+1}(t_1, \ldots, t_{n+1}) -$$

$$- \int_{X^{n+1}} f\left(\frac{\bar{t}_1 + \ldots + \bar{t}_{n+1}}{n+1}\right) d\mu_{x,n+1}(t_1, \ldots, t_{n+1}).$$

Hence, for all $f \in C(X)$,

(5) $$B_n f(x) - B_{n+1} f(x) = \int_{X^{n+1}} \left[\frac{f(\bar{t}_1) + \ldots + f(\bar{t}_{n+1})}{n+1} - f\left(\frac{\bar{t}_1 + \ldots + \bar{t}_{n+1}}{n+1}\right)\right] \cdot d\mu_{x,n+1}(t_1, \ldots, t_{n+1}).$$

If $f \in C(X)$ is convex it follows that $B_n f(x) - B_{n+1} f(x) \geq 0$ and the proof is complete.

REMARK 1. Another proof of this theorem, similar to that of Theorem 2 in [13] can be given; it suffices to observe that B_n satisfies a relation similar to (9) in [13].

REMARK 2. The formulas (1.3) in [1] and the similar ones given in [6, Sect. 3] can be derived from (5).

COROLLARY. *Let $f \in C(X)$ be a convex function and let $x \in X$. The following statements are equivalent*:

(i) *f is affine on* $\mathrm{cl}(\mathrm{conv}(\mathrm{supp}\, \mu_x))$

(ii) $\mu_x(f) = f(x)$

(iii) $B_n f(x) = B_{n+1} f(x)$ *for all* $n \geq 1$.

(iv) *For all* $n \geq 1$ *and all* $(t_1, \ldots, t_{n+1}) \in (\mathrm{supp}\, \mu_x)^{n+1}$,

$$f\left(\frac{\bar{t}_1 + \ldots + \bar{t}_{n+1}}{n+1}\right) = \frac{f(\bar{t}_1) + \ldots + f(\bar{t}_{n+1})}{n+1}.$$

Proof. (i) $\Leftrightarrow$ (ii). This is Theorem 1 in [8].

(ii) $\Leftrightarrow$ (iii). Use (1) and the following consequence of (2) and (4):

$$\mu_x(f) = B_1 f(x) \geq B_2 f(x) \geq \ldots \geq f(x).$$

(iii) $\Leftrightarrow$ (iv). It suffices to apply (5) and the equality $\mathrm{supp}\,(\mu_{x,n+1}) = (\mathrm{supp}\, \mu_x)^{n+1}$.

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