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A NEW CLASS OF LINEAR POSITIVE OPERATORS OF BERNSTEIN TYPE

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Sommario. In questo lavoro si introduce e si studia una nuova classe di operatori lineari e positivi di tipo Bernstein. Si evidenziano numerose proprietà e si dimostrano alcuni teoremi di convergenza.

Abstract. In this paper a new class of linear positive operators of Bernstein type is introduced and studied. Several properties and some convergence theorems are given.

1. Introduction. Let f be a continuous function on $I = [0, 1]$ ($f \in C^0(I)$) and denote by $(S_n^a f)(x)$ the corresponding Stancu polynomial of degree n. It is well known that

$$(S_n^a f)(x) = S_n^a(f; x) = \sum_0^n {}_k \frac{p_{n,k}^a(x)}{1^{(n,-a)}} f\left(\frac{k}{n}\right), \quad n \in N \text{ and } \alpha \in R^+$$

where

$$p_{n,k}^a(x) = \binom{n}{k} x^{(k,-a)}(1-x)^{(n-k,-a)}, \quad x \in I$$

and

$$x^{(k,-a)} = x(x+a)\dots(x+(k-1)a).$$

This operator was introduced in [16] and studied in [9—11, 17—20].

Moreover, from S_n^a operator one can obtain [17], as limiting cases, the following two operators:

1) Favard-Szasz-Mirakyan operator M_n

$$M_n(f; x) = e^{-nx} \sum_0^\infty {}_k \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

with $x \geqslant 0$ and $|f(x)| = 0(x^{\beta x})$, where β is a positive arbitrary fixed number [4, 7, 13, 22, 24].

2) Baskakov operator P_n

$$P_n(f;x) = \sum_{0}^{\infty}{}_k \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right)$$

with $x \geqslant 0$ and $|f(x)| = 0(e^{\beta x})$, where β is a positive arbitrary fixed number [1, 6, 23].

It is well known that S_n^a is a linear positive operator verifying the following relation [11]:

$$\lim_n \frac{d^p}{dx^p} S_n^a(f;x) = f^{(p)}(x), \quad n > p \geqslant 0$$

uniformly on I, $\forall\, a = a_n = 0(n^{-2})$ and $\forall f \in C^p(I)$.

In [10] Mastroianni and Occorsio introduced and studied the operator:

$$S_{n,k}^a(f;x) = [I-(I-S_n^a)]^k(f;x) = \sum_{1}^{k}{}_i (-1)^{i-1}\binom{k}{i}(S_n^a)^i(f;x)$$

where k is an integer number greater than or equal to 1 and $(S_n^a)^i$ is the i-th iterate of S_n^a operator defined by

$$(S_n^a)^1 = S_n^a \text{ and } \forall\, i > 1 \ (S_n^a)^i = S_n^a(S_n^a)^{i-1}, \ ((S_n^a)^0 = I).$$

$S_{n,k}^a$ is not positive in general, but, when f is sufficiently smooth, $S_{n,k}^a$ approximates the function f better than $S_n f$ [10].

The aim of this paper is to study the following more general operator

(1.1) $$S_{n,\lambda}^a(f;x) = [I-(I-S_n^a)]^\lambda(f;x) = \sum_{1}^{\infty}{}_i (-1)^{i-1}\binom{\lambda}{i}(S_n^a)^i(f;x)$$

with $f \in C^0(I)$ and $a \in R^+$.

We point out many properties of $S_{n,\lambda}^a$ (Sections 2 and 3). Finally, in Section 4 we study the convergence for $m \to \infty$ and $\lambda \to \infty$, separately.

2. General properties of $S_{n,\lambda}^a$ operator. $S_{n,\lambda}^a$ operator defined by (1.1), for $a = 0$, coincides with $B_{n,\lambda}$ operator, introduced and studied by Mastroianni and Occorsio in [12].

In particular we have

$$S_{n,1}^a = S_n^a, \quad S_{1,\lambda}^a = S_1^a, \quad S_{n,\lambda}^a(f;0) = f(0), \quad S_{n,\lambda}^a(f;1) = f(1)$$

and

(2.1) $$S_{n,\lambda}^a(e_i) = e_i, \quad i = 0, 1, \quad (e_k(x) = x^k, \ k \in N).$$

So $S_{n,\lambda}^a$ has exactness degree 1 and interpolates the function f in 0 and 1.

We denote now by $L_n f$ the Lagrange polynomial interpolating the function f on the knots $\frac{i}{n}$, $i = 0, 1, \ldots, n$.

Being

$$S_n^{-1/n} f = L_n f,$$

from (1.1), we have also

$$S_n^{-1/n} f = L_n f, \quad \forall\, \lambda > 0.$$

Finally, again from the definition, it follows that, if $\lambda \in (0, 1]$, $S_{n,\lambda}^a$ is a positive operator.

Now we want to represent $S_{n,\lambda}^a$ ($\forall\, \lambda \in R^+$) in a matrix form. We denote by S_i^j and s_i^j the Stirling numbers of first and second kind, respectively, defined by

$$x^{(n,h)} = \sum_{0}^{n-1}{}_j S_j^n x^{n-j} h^j \text{ and } x^n = \sum_{0}^{n-1}{}_j s_j^n x^{(n-j,h)} h^j, \ \forall\, h \in R \text{ and } n \geqslant 1$$

One can verify easily that, $\forall\, k \geqslant 1$, it is

(2.2) $$\left[0, \frac{1}{n}, \ldots, \frac{l}{n}; e_k\right] = \frac{s_{k-1}^k}{n^{k-1}}$$

where $\left[0, \frac{1}{n}, \ldots, \frac{l}{n}; e_k\right]$ is the divided difference of order l, of the function e_k, with respect to the nodes $0, \frac{1}{n}, \ldots, \frac{1}{n}$. We recall now that $S_n f$ can be expressed in the following form [16]

(2.3) $$S_n^a(f;x) = \sum_{0}^{n}{}_k \gamma_{n,k}^a x^{(k,-a)}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right]$$

with

$$\gamma_{n,k}^a = \frac{1^{(k,1/n)}}{1^{(k,-a)}}, \quad \forall\, k \in N, \ (\gamma_{n,0}^a = 1).$$

From this, by (2.2), it follows

(2.4) $$S_n^a(e_k;x) = \sum_{1}^{n}{}_l \gamma_{n,l}^a x^{(l,-a)} s_{k-1}^k n^{1-k}$$

Then, letting

$$h_{k,-a} = (x, x^{(2,-a)}, \ldots, x^{(k,-a)}), \ u_k = (0, 0, \ldots, 1)^T \in R^k,$$

$$\Gamma_{k,-a} = \begin{pmatrix} \gamma_{n,1}^a & & 0 \\ & \gamma_{n,2}^a & \\ 0 & & \gamma_{n,k}^a \end{pmatrix}$$

$$T_{k,h} = \begin{pmatrix} h^k & s_k^{k+1} \\ h^{k-1} & s_{k-1}^{k+1} \\ \cdot \quad \cdot & \cdot \quad \cdot \\ h^2 & s_2^{k+1} \\ h & s_1^{k+1} \end{pmatrix}$$

and

$$D_{k,h} = \begin{cases} 1, & k = 0 \qquad \forall\, h \in R \\ \begin{pmatrix} D_{k-1,h} & T_{k,h} \\ \underline{0} & 1 \end{pmatrix}, & k > 0, \quad (\underline{0} \in R^k) \end{cases}$$

(2.4) can be written as follows

$$S_n^a(e_k\,;\,x) = h_{k,-a}\Gamma_{k,a}D_{k-1,1/n}u_k$$

Now, letting

$$\eta_{k,-a} = x^{(k,-a)}$$

and following [10], we obtain

$$(2.5) \qquad (S_n)^i(\eta_{k,-a}\,;\,x) = h_{k,-a}(\Gamma_{k,a}D_{k-1,1/n}D^{-1}_{k-1,-a})^i u_k = \\ = h_{k,-a}N^i_{k,1/n,-a}u_k \qquad (\forall\, i \in N)$$

The matrix

$$(2.6) \qquad N_{k,1/n,-a} = \Gamma_{k,a}D_{k-1,1/n}D^{-1}_{k-1,-a}$$

has as eigenvalues the numbers

$$\gamma^a_{n,i} = \frac{1^{(i,1/n)}}{1^{(i,-a)}} \leqslant 1, \qquad (i = 1, 2, \ldots, k).$$

Then, we denote by $W_{k,n,a}$ the upper triangular matrix, having as columns the eigenvectors of $N_{k,1/n,-a}$, normalized so that the elements of the main diagonal are 1. So it results

$$(2.7) \qquad N_{k,1/n,-a} = W_{k,n,a}\Gamma_{k,a}W^{-1}_{k,n,a}$$

Therefore (2.5) can be written as follows

$$(2.8) \qquad (S_n)^i(\eta_{k,-a}\,;\,x) = h_{k,-a}W_{k,n,a}\Gamma^i_{k,a}W^{-1}_{k,n,a}u_k$$

From (2.8) and being

$$(S_n^a)^i(f\,;\,x) = \sum_0^n{}_k\, \gamma^a_{n,k}(S_n^a)^{i-1}(\eta_{k,-a}\,;\,x)\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right]$$

we obtain

$$(2.9) \qquad (S_n^a)^i(f\,;\,x) = \sum_0^n{}_k\, \gamma^a_{n,k}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] \times \\ \times \{h_{k,-a}W_{k,n,a}\Gamma^{i-1}_{k\,a}W^{-1}_{k,n,a}u_k\}$$

By this relation, (1.1) becomes

$$(2.10) \qquad S_{n,\lambda}(f;\,x) = \sum_1^\infty{}_i\, (-1)^{i-1}\binom{\lambda}{i}\left\{\sum_0^n{}_k\, \gamma^a_{n,k}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] \times \right. \\ \left. \times\, h_{k,-a}W_{k,n,a}\Gamma^{i-1}_{k,a}W^{-1}_{k,n,a}u_k\right\} = \sum_0^n{}_k\, \gamma^a_{n,k}h_{k,-a}W_{k,n,a}\left\{\sum_0^\infty{}_j\, (-1)^j\binom{\lambda}{j+1}\Gamma^j_{k,a}\right\} \times \\ \times\, W^{-1}_{k,n,a}u_k\left[0, \frac{1}{n}, \ldots, \frac{1}{n}; f\right].$$

On the other hand, being

$$(2.11) \qquad \sum_0^\infty{}_j\, (-1)^j\binom{\lambda}{j+1}t^j = t^{-1}[1-(1-t)^\lambda] = \psi(t, \lambda), \quad \forall\, t \in (0, 1]$$

and

$$\gamma^a_{n,k} \leqslant 1, \quad k = 0, 1, \ldots, n,$$

(2.10) can be written as follows

$$(2.12) \qquad S^a_{n,\lambda}(f\,;x) = \sum_0^n{}_k\, \gamma^a_{n,k}z^a_{n,k}(x, \lambda)\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right]$$

where

$$z^a_{n,k}(x, \lambda) = h_{k,-a}W_{k,n,a}F_{k,a,\lambda}W^{-1}_{k,n,a}u_k$$

and

$$F_{k,a,\lambda} = \sum_0^\infty{}_j(-1)^j\binom{\lambda}{j+1}(\Gamma_{k,a})^j$$

is a diagonal matrix whose elements are $\psi(\gamma^a_{n,i}, \lambda)$, $i = 1, 2, \ldots, k$. We notice that $z^a_{n,k}$ is a polynomial in x of degree not greater than k. Therefore, from (2.12), it follows that $S^a_{n,\lambda}f$ is a polynomial of degree not greater than n. Moreover, if f is a polynomial of degree not greater than $p (p \leqslant n)$, then also $S^a_{n,\lambda}f$ is.

Recalling (1.1), (2.12) can also be written as follows

$$(2.13) \qquad S^a_{n,\lambda}(f;\,x) = \sum_0^n{}_k\, p^a_{n,k}(x)\, y\left(\frac{k}{n}\right)[1^{(n,-a)}]^{-1}$$

where

$$(2.14) \qquad y(x) = \sum_0^\infty{}_j(-1)^j\binom{\lambda}{j+1}(S^a_n)^j\,(f\,;\,x)$$

Now we denote by $R^a_{n,\lambda}$ the remainder of $S^a_{n,\lambda}$ operator, defined by

$$R^a_{n,\lambda} = I - S^a_{n,\lambda}$$

So we have

$$(2.15)\qquad R^a_{n,\lambda} = (I - S^a_n)^\lambda = \sum_{0}^{\infty}{}_i (-1)^i \binom{\lambda}{i} (S^a_n)^i$$

In particular, by (2.8), it follows

$$R^a_{n,\lambda}(\eta_{k,-a}\,;\,x) = h_{k,-a} W_{k,n,a} M_{k,a,\lambda} W^{-1}_{k,n,a} u_k$$

where $M_{k,a,\lambda}$ is the diagonal matrix whose elements are $(1-\gamma^a_{n,i})^\lambda$, $i = 1, 2, \ldots, k$.

The procedure given above can be used really to evaluate $S^a_{n,\lambda} f$. For example, we have

$$(2.16)\qquad S^a_{n,\lambda}(e_2\,;\,x) = x^2 + x(1-x)\left[\frac{n\alpha+1}{n(1+\alpha)}\right]^\lambda,$$

$$R^a_{n,\lambda}(e_2\,;\,x) = -S^a_{n,\lambda}((t-x)^2\,;\,x) = x(1-x)\left[\frac{n\alpha+1}{n(1+\alpha)}\right]^\lambda,$$

$$S^a_{2,\lambda}(f\,;\,x) = 2(1-\vartheta^\lambda)(f_0 - 2f_{1/2} + f_1)\,x^2 + [(2\vartheta^\lambda - 3)f_0 + 4(1-\vartheta^\lambda)f_{1/2} + (2\vartheta^\lambda - 1)f_1]\,x + f_0,$$

where

$$\vartheta = (2a+1)(2a+2)^{-1}.$$

Finally we notice that $S^a_{n,\lambda}$ operator $(\lambda \in (0,1))$ is a particular case of the operator

$$\bar{S}^a_{n,m,\lambda}(f\,;\,x) = \sum_{m+1}^{\infty}{}_i (-1)^{m+1+i}\binom{\lambda}{i}(S^a_n)^i(f\,;\,x), \quad m < \lambda < m+1$$

However, it is easy to verify that, $\forall\, m > 0$, $\bar{S}^a_{n,m,\lambda} f$ does not converge to f. So $S^a_{n,\lambda}$ is the only operator of the class $\{\bar{S}^a_{n,m,\lambda}\}_{m\in N^0}$ converging to f.

3. The properties of $S^a_{n,\lambda}$ operator $(\lambda \in (0,1))$. The following propositions hold:

PROPOSITION 3.1. *$S^a_{n,\lambda}$ operator $(\lambda \in (0,1))$ preserves the concavity (convexity) of every order of the function to approximate.*

Proof. Indeed, we recall [11] that S^a_n, and therefore its iterates, preserve the concavity or the convexity of every order s $(s \in N)$ of the function f to approximate.

So, from (2.14), if $\lambda \in (0,1)$, also the function y is concave (convex) of order s, and then, by (2.13), $S^a_{n,\lambda} f$ is concave (convex) of order s.

PROPOSITION 3.2. *$S^a_{n,\lambda}$ operator $(\lambda \in (0,1))$ verifies the following relation, $\forall\, x \in (0,1)$,*

$$S^a_{n+1,\lambda}(f\,;\,x) - S^a_{n,\lambda}(f\,;\,x) = x(1-x)(1+a)^{-\lambda}\left\{\left[\frac{(n+1)a+1}{n+1}\right]^\lambda - \left[\frac{na+1}{n}\right]^\lambda\right\}[\bar{x}^1, \bar{x}_2, \bar{x}_3\,;\,f], \quad \forall f \in C^\circ(I) \qquad (3.1)$$

where $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are suitable points of $(0,1)$ generally depending on f.

Proof. Recalling the expression of $(S^a_{n+1} - S^a_n)(f\,;\,x)$ [16], from (2.13) we have

$$(S^a_{n+1,\lambda} - S^a_{n,\lambda})(f\,;\,x) = -\frac{1}{n(n+1)}\sum_{0}^{n}{}_k\left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}\right]\binom{n-1}{k}\times$$
$$\times\frac{x^{(k+1,-a)}(1-x)^{(n-k,-a)}}{1^{(n+1,-a)}}$$

So $S^a_{n+1,\lambda} - S^a_{n,\lambda}$ has exactness degree 1. Moreover, as noticed before if f is convex of first order and $\lambda \in (0,1)$, then also y is convex of firs, order; therefore $(S^a_{n+1,\lambda} - S^a_{n,\lambda})(f\,;\,x) \neq 0$ $\forall\, x \in (0,1)$. Then, by a theorem of Popoviciu [15], we can find three points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3 \in (0,1)$ such that, $\forall\, x \in (0,1)$,

$$(S^a_{n+1,\lambda} - S^a_{n,\lambda})(f\,;\,x) = (S^a_{n+1,\lambda} - S^a_{n,\lambda})(e_2\,;\,x)[\bar{x}_1, \bar{x}_2, \bar{x}_3\,;\,f], \quad \forall f \in C^\circ(I)$$

Recalling (2.16), (3.1) follows. From (3.1), we obtain

COROLLARY 3.3. *If the function f is convex (concave) of first order, then, $\forall\, x \in (0,1)$*

$$S^a_{n+1,\lambda}(f\,;\,x) \underset{(>)}{<} S^a_{n,\lambda}(f\,;\,x).$$

We notice that, if $a = a_n = 0(1)$ $(n \to \infty)$ and if $\lambda \in (0,1)$, then, by (2.1) and (2.16), $S^a_{n,\lambda}$ verifies the hypotheses of Korovkin's theorem and therefore $S^a_{n,\lambda} f$ converges uniformly to f on I.

From this and by a theorem proved by Moldovan in [14], we obtain

COROLLARY 3.4. *If f is a continuous function on I and if the sequence $\{S^a_{n,\lambda}(f\,;\,x)\}_n$ $(\lambda \in (0,1))$ is not-decreasing (not-increasing), then the function f is not-convex (not-concave) of first order on I.*

Corollary 3.4, together with the previous Corollary 3.3, gives a characterization of functions convex (concave) of first order on I.

Now we denote by $\mathrm{Lip}_M\mu$ the class of Hölder-continuous functions, i.e.

$$\mathrm{Lip}_M\mu = \{f \in C^\circ(I) / |f(x) - f(y)| \leqslant M|x-y|^\mu\}.$$

The following proposition holds:

PROPOSITION 3.5. *If* $a = a_n = 0(1)$ $(n \to \infty)$, *then*

$$(3.2) \qquad f \in \mathrm{Lip}_M\mu \Leftrightarrow S^a_{n,\lambda} f \in \mathrm{Lip}_M \mu, \quad \forall\, n \geqq 1 \text{ and } \forall\, \lambda \in (0,1].$$

Proof. Let f be a function $\in \mathrm{Lip}_M\mu$ and x and y any two points on I. Then, by a theorem proved in [3], also the j-th iterate of Stancu operator $(\forall\, j \in N) \in \mathrm{Lip}_M\mu$, i.e.

$$(3.3) \quad |f(x) - f(y)| \leqslant M|x-y|^\mu \Rightarrow |(S^a_n)^j (f;x) - (S^a_n)^j (f;x)| \leqslant M|x-y|^\mu$$

Therefore, from the definition of $S^a_{n,\lambda}$ operator, by (3.3) it follows

$$|S^a_{n,\lambda}(f;x) - S^a_{n,\lambda}(f;y)| = \left| \sum_1^\infty {}_j (-1)^j \binom{\lambda}{j} \{(S^a_n)^j (f;x) - (S^a_n)^j (f;y)\} \right| \leqslant$$

$$\leqslant (2^\lambda - 1) M |x-y|^\mu \leqslant M|x-y|^\mu$$

We notice that, if $a = a_n = 0(1)$, $(n \to \infty)$ and $\lambda \in (0,1]$, then, as observed before, $S^a_{n,\lambda} f$ converges uniformly to f on I; so, in these hypotheses, the converse of (3.2) holds too. A simple expression of the remainder $R^a_{n,\lambda} f$ can also be given. In fact, because $S^a_{n,\lambda}$ $(\lambda \in (0,1])$ is a positive operator with exactness degree 1, then, by a theorem proved in [8], we have

$$(3.4) \quad R^a_{n,\lambda}(f;x)) = \frac{1}{2} R^a_{n,\lambda}(e_2;x) f''(\xi) = -\frac{x(1-x)}{2}\left[\frac{na+1}{n(1+a)}\right]^\lambda f''(\xi)$$

$\forall\, f \in C^2(I)$ and with ξ a suitable point in $(0,1)$.
Now, letting

$$D_a g(x) = [g(x+a) - g(x)]\, a^{-1}, \quad D^m_a = D_a(D^{m-1}_a), \quad (D^0_a g(x) = g(x)),$$

$\forall\, x \in I$, $\forall\, \alpha \in R^+$ and $\forall\, g$ defined on I, the following proposition holds:

PROPOSITION 3.6. *The sequence*

$$(3.5) \qquad \{D^m_a S_{n,\lambda}(f;x)\}_n, \quad 0 < ma < 1, \quad m < n, \quad \lambda \in (0,1] \text{ and } x \in I$$

verifies the following monotonicity properties:

i) *for* $m = 1$

a) *if on the interval* $\left[0, \frac{1-a}{2}\right]$ *the function* f *is convex (concave) of first and second order, then the sequence* (3.5) *is decreasing (increasing) on* $\left[0, \frac{1-a}{2}\right]$;

b) *if on the interval* $\left[\frac{1-a}{2}, 1-a\right]$ *the function* f *is concave (convex) of first order and convex of first order and convex (concave) of second order, then the sequence* (3.5) *is decreasing (increasing) on* $\left[\frac{1-a}{2}, 1-a\right]$;

c) *if on the interval* $[1-a, 1]$ *the function* f *is concave (convex) of first and second order, then the sequence* (3.5) *is decreasing (increasing) on* $[1-a, 1]$;

ii) *for* $m \geqslant 2$

a) *if on the interval* $\left[0, \frac{1-(2m-1)a}{2}\right]$ *the function* f *is concave (convex) of order* $m-1$ *and convex (concave) of order* m *and* $m+1$, *then the sequence* (3.5) *is decreasing (increasing) on* $\left[0, \frac{1-(2m-1)a}{2}\right]$;

b) *if on the interval* $\left[\frac{1-(2m-1)a}{2}, 1-ma\right]$ *the function* f *is concave (convex) of order* $m-1$ *and* m *and convex (concave) of order* $m+1$, *then the sequence* (3.5) *is decreasing (increasing) on* $\left[\frac{1-(2m-1)a}{2}, 1-ma\right]$;

c) *if on the interval* $[1-ma, 1]$ *the function* f *is concave (convex) of order* $m-1$, m *and* $m+1$, *then the sequence* (3.5) *is decreasing (increasing) on* $[1-ma, 1]$.

Proof. Proposition 3.6 follows from (2.13) and from a theorem proved in [2].

Remark. We notice that, for $a = 0$, Proposition 3.6 gives a new monotonicity result for the sequence

$$\{(B_{n,\lambda} f)^{(m)}(x)\}_n, \quad \lambda \in (0,1], \quad m < n \text{ and } x \in I.$$

This result is more general than that one obtained by Stancu in [21] for the sequence $\{(B_n f)^{(m)}(x)\}_n$. Finally we prove

PROPOSITION 3.7. *Let* $f(x)$ *be a piecewise linear function with at most* $n-1$ *changes of slope, which can occur only at the points* $\frac{i}{n}$, $i = 1, 2, \ldots, n-1$. *Then, for all natural numbers* m, $S^a_{nm+1,\lambda}(f;x)$ *is of degree* nm *and, moreover,* $S^a_{nm,\lambda}(f;x) = S^a_{nm+1,\lambda}(f;x)$.

Proof. Proposition 3.7 follows from (1.1) and from a theorem proved by Freedman and Passow in [5].

4. On the convergence of $S^a_{n,\lambda}$. Assume that $\lambda \in (0,1)$ and $a = a_n = 0(1)$ $(n \to \infty)$. Then, as noticed before, $S^a_{n,\lambda}$ verifies the hypotheses of Korovkin's theorem on I and $S^a_{n,\lambda} f$ converges to f uniformly on I, $\forall f \in C^0(I)$. Moreover, following [12], we obtain the evaluations:

$$(4.1) \qquad \|f - S^a_{n,\lambda} f\| \leqslant \frac{5}{4}\, \omega\left(f; \left[\frac{na+1}{n(1+a)}\right]^{\lambda/2}\right), \quad \forall\, f \in C^0(I)$$

$$(4.2) \quad \|f - S^a_{n,\lambda} f\| \leqslant \frac{3}{4}\left[\frac{na+1}{n(1+a)}\right]^{\lambda/2}\Bigg)\, \omega\left(f'; \left[\frac{na+1}{n(1+a)}\right]^{\lambda/2}\right), \quad \forall\, f' \in C^0(I).$$

and the punctual estimates, $\forall\, x \in I$,

$$|f(x) - S^a_{n,\lambda}(f;\, x)| \leqslant 2\omega\left(f;\, \sqrt{x(1-x)\left[\frac{na+1}{n(1+a)}\right]^\lambda}\right),\ \forall f \in C^0(I)$$

$$|f(x) - S^a_{n,\lambda}(f;\, x)| \leqslant 2\sqrt{x(1-x)\left[\frac{na+1}{n(1+a)}\right]^\lambda}\ \omega\left(f';\, \sqrt{x(1-x)\left[\frac{na+1}{n(1+a)}\right]^\lambda}\right),$$

$$\forall f' \in C^0(I)$$

We notice that, for $a = 0$ in (4.1) and (4.2), we find the relations obtained by Mastroianni and Occorsio in [12] for $B_{n,\lambda}$ operator.

If λ is an integer greater than 1, then $S^a_{n,\lambda}$ was studied by Mastroianni and Occorsio in [10]. So assume that $\lambda = i + \delta$, with i biggest integer number $< \lambda$ and $\delta \in (0, 1)$. Then, from (3.4), it follows:

$$|R^a_{n,\lambda}(f;\, x)| = |R^a_{n,\delta}(R^a_{n,i}(f;\, x))| \leqslant \frac{1}{8}\, \|(R^a_{n,i}f)^{(2)}\| \left[\frac{na+1}{n(1+a)}\right]^\delta$$

On the other hand, because [10]

$$\|(R^a_{n,i}f)^{(2)}\| \leqslant C(n-2)^{-i}\, \|f^{(2)}\|_{2i}, \quad \forall f \in C^{2i+2}(I)$$

where

$$\|f\|_{2i} = \max_{0 \leqslant j \leqslant 2i} \|f^{(j)}\|$$

and C is a constant depending on i and independent of f and n, we obtain

COROLLARY 4.2. *Let $\lambda = i + \delta$, with $i \in N$ and $\delta \in (0, 1)$. Then, $\forall f \in C^{2i+2}(I)$, it results*

$$\|R^a_{n,\lambda}f\| \leqslant \sigma_i\, \|f^{(2)}\|_{2i} \left[\frac{na+1}{1+a}\right]^\delta n^{-i-\delta}, \quad n > 2$$

where σ_i is a constant independent of f and n.

Finally we prove

THEOREM 4.3. *Let $f \in C^0(I)$. Then, the following relation holds:*

$$\lim_{\lambda\to\infty} S^a_{n,\lambda}(f;\, x) = L_n(f;\, x) \tag{4.5}$$

where $L_n(f;\, x)$ is Lagrange polynomial interpolating the function f on the knots $\frac{i}{n}$, $i = 0, 1, \ldots, n$.

Proof. First of all, we notice that, from (2.12), it follows

$$\lim_{\lambda\to\infty} S^a_{n,\lambda}(f;\, x) = \sum_0^n {}_k\, \gamma^a_{n,k} \frac{n^k}{k!} h_{k,-a} W_{k,n,a} [\lim_{\lambda\to 8} F_{k,a,\lambda}] \cdot W^{-1}_{k,n,a} u_k \Delta^k_{1/n}$$

From (2.11), recalling that the eigenvalues $\gamma^a_{n,i}$, $i = 1, 2, \ldots, k$, are not greater than 1, we have

$$\lim_{\lambda\to\infty} S^a_{n,\lambda}(f;\, x) = \sum_0^n {}_k\, \gamma^a_{n,k} \frac{n^k}{k!} h_{k,-a} W_{k,n,a} \Gamma^{-1}_{k,a} W^{-1}_{k,n,a} u_k \Delta^k_{1/n} f(0) =$$

$$= \sum_0^n {}_k\, \gamma^a_{n,k} \frac{n^k}{k!} h_{k,-a} W_{k,n,a} (W_{k,n,a}\Gamma_{k,a})^{-1} u_k \Delta^k_{1/n} f(0)$$

On the other hand, from (2.7), it follows

$$h_{k,-a} W_{k,n,a} (W_{k,n,a}\Gamma_{k,a})^{-1} = h_{k,-a} W_{k,n,a} (N_{k,1/n,a} W_{k,n,a})^{-1} =$$

$$= h_{k,-a} W_{k,n,a} W^{-1}_{k,n,a} D_{k-1,-a} D^{-1}_{k-1,1/n} \Gamma^{-1}_{k,a}$$

and letting

$$X_k = (x, x^2, \ldots, x^k),$$

it results

$$h_{k,-a} W_{k,n,a} (W_{k,n,a}\Gamma_{k,a})^{-1} = X_k D^{-1}_{k-1,1/n} \Gamma^{-1}_{k,a} = h_{k,1/n} \Gamma^{-1}_{k,a}$$

So we can write

$$\lim_{\lambda\to\infty} S^a_{n,\lambda}(f;\, x) = \sum_0^n {}_k\, \gamma^a_{n,k} \frac{n^k}{k!} h_{k,1/n} \Gamma^{-1}_{k,a} u_k \Delta^k_{1/n} f(0) =$$

$$= \sum_0^n {}_k \frac{n^k}{k!}\, x^{(k,1/n)} \Delta^k_{1/n} f(0)$$

that is (4.5).

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