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REPRESENTATION OF CONTINUOUS LINEAR FUNCTIONALS ON SMOOTH NORMED LINEAR SPACES

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Abstract. In this paper we shall give some theorems of representation for the continuous linear functionals on smooth normed linear spaces by use of the semi-inner product in the sense of Lumer [3] and Tapia [5], and the best approximation in normed linear spaces by elements of linear subspaces.

Introduction. DEFINITION 1 ([3], [1] pp. 389). Let X be a real or complex linear space. A mapping $(\ ,\ )_L : X \times X \to K(\mathbb{R}, \mathbb{C})$ is called semi-inner product in the sense of Lumer or L-semi-inner product, for short, if the following conditions hold :

(i) $(x + y, z)_L = (x, z)_L + (y, z)_L, \quad x, y, z \in X$;

(ii) $(\alpha x, y)_L = \alpha(x, y)_L, \quad \alpha \in K, \quad x, y \in X$;

(iii) $(x, x)_L > 0$ if $x \neq 0$;

(iv) $|(x, y)_L|^2 \leqslant (x, x)_L (y, y)_L, \quad x, y \in X$;

(v) $(x, \lambda y)_L = \bar{\lambda}(x, y)_L, \quad \lambda \in K, \quad x, y \in X$.

For the properties of L-semi-inner product, we send to [1] pp. 386–389, or [2] where further references are given.

DEFINITION 2 ([5], [1] pp. 389). Let $(X, \|\cdot\|)$ be a real normed linear space and $f : X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$.

Then the mapping :

$$(x, y)_T = (V_+ f)(y) \cdot x = \lim_{t \downarrow 0} \frac{f(y + tx) - f(y)}{t}, \quad x, y \in X ;$$

is called semi-inner product in the sense of Tapia or T-semi-inner product, for short.

For the usual properties of T-semi-inner product, we send to [1] pp. 389–393 or [2] where further references are given.

In paper [2] we proved the following results:

LEMMA A. *Let $(X, \|\cdot\|)$ be a normed linear space and $(\ ,\ )_{\mathrm{L}}$ a L-semi-inner product which generates the norm $\|\cdot\|$. Then the following conditions are equivalent:*

(i) $(X, \|\cdot\|)$ *is a smooth normed linear space;*

(ii) *for every $x, y \in X$ there exist the limits*

$$\lim_{t\to 0} \operatorname{Re}(y, x+ty)_{\mathrm{L}} \text{ and } \lim_{t\to 0} \frac{\operatorname{Re}(x, x+ty)_{\mathrm{L}} - (x, x)_{\mathrm{L}}}{t}.$$

LEMMA B. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then we have:*

$$(1)\qquad (y, x)_T = \operatorname{Re}(y, x)_{\mathrm{L}} = \lim_{t\to 0} \frac{\operatorname{Re}(x, x+ty)_{\mathrm{L}} - (x, x)_{\mathrm{L}}}{t}$$

for all $x, y \in X$.

The following lemma of L-orthogonality holds:

LEMMA C. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. If for every $\lambda \in K$, we have:*

$$(2)\qquad \|x + \lambda y\| \geq \|x\|,$$

then

$$(3)\qquad x \mathrm{L}\, y \text{ i.e. } (y, x)_{\mathrm{L}} = 0.$$

Using the above lemmas, we proved the following two theorems of representation:

THEOREM D. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then for every E a closed linear subspace in X and for all $x \in X$, there exists $x' \in E$ and $x'' \in E^{\mathrm{L}}$ such that*

$$(4)\qquad x = x' + x'',$$

where E^{L} denote the orthogonal complement in the sense of Lumer of E i.e. the set $\{y \in X \mid y \mathrm{L}\, x, \text{ for all } x \in E\}$.

THEOREM E. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then for every $f \in X^*$ there exists an element $u_f \in X$ such that:*

$$(5)\qquad f(x) = (x, u_f)_{\mathrm{L}}, \quad \|f\| = \|u_f\|, \quad x \in X.$$

In addition, if $f \neq 0$, then the representation element u_f is given by:

$$(6)\qquad u_f = \frac{\overline{f(w)}}{\|w\|^2}\, w$$

where $w \in \operatorname{Ker}(f)^{\mathrm{L}}$ and $w \neq 0$.

COROLLARY 1. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space over the complex number field. Then for every $f \in X^*$, there exists an element u_f such that:*

$$(7)\qquad f(x) = (x, u_f)_T - \mathrm{i}(\mathrm{i}x, u_f)_T, \quad \|f\| = \|u_f\|, \ x \in X.$$

COROLLARY 2. *Let $(X, \|\cdot\|)$ be a smooth reflexive Banach space over the complex number field and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then for every $f \in X^*$ there exists an element $u_f \in X$ such that:*

$$(8)\qquad f(x) = \lim_{t\to 0} \frac{\operatorname{Re}(u_f, u_f + tx)_{\mathrm{L}} - \|u_f\|^2}{t} - \mathrm{i} \lim_{t\to 0} \frac{\operatorname{Re}(u_f, u_f + \mathrm{i}tx)_{\mathrm{L}} - \|u_f\|^2}{t}$$

for all $x \in X$ and $\|f\| = \|u_f\|$.

DEFINITION 3. The element $x \in X$ is called orthogonal in the sense of Birkhoff over $y \in X$ iff $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in K$. We note that $x \perp y$.

Now, let G be a proper linear subspace not dense in X and

$$(9)\qquad \mathscr{P}_G(x_0) := \{y_0 \mid \|y_0 - x_0\| = \inf_{y\in G} \|y - x_0\|\} \subset G,$$

the set of all elements of best approximation referring to $x_0 \in X \setminus \overline{G}$. The following lemma of characterization in terms of Birkhoff's orthogonality holds:

LEMMA F. ([4] pp. 85). *Let $(X, \|\cdot\|)$ be a normed linear space, G a linear subspace in X, $x_0 \in X \setminus \overline{G}$ and $g_0 \in G$. Then $g_0 \in \mathscr{P}_G(x_0)$ iff $x_0 - g_0 \perp G$.*

DEFINITION 4. The proper linear subspace $E \subset X$ is called proximinal in X iff for every $x \in X$ the set $\mathscr{P}_G(x)$ is nonvoid.

LEMMA G. ([4] pp. 87). *Let $(X, \|\cdot\|)$ be a normed linear space and H a hyperplane in X such that $0 \in H$. Then H is proximinal in X iff there exists $z \in X \setminus \{0\}$ such that $z \perp H$.*

For details concerning the theory of elements of best approximation in normed linear spaces, we send to monograph [4] due to Ivan Singer.

1. Representation theorems. We shall begin our considerations by the following lemma which completes Lemma C of L-orthogonality.

1.1. LEMMA. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and $(\ ,\ )_{\mathrm{L}}$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then the following conditions are equivalent:*

(i) $x \perp y$;

(ii) $x \mathrm{L}\, y$.

Proof. The implication "(i) ⇒ (ii)" follows by Lemma C. "(ii) ⇒ (i)". We shall use the following result due to R.C. James (see [4] pp. 85):

THEOREM (R. C. James). *Let $(X, \|\cdot\|)$ be a real normed linear space. Then the following conditions are equivalent:*

(i) $x \perp \alpha x + y$;

(ii) $-\tau(x, -y) \leqslant \alpha \|x\| \leqslant \tau(x, y)$;

where τ is given by:

$$\tau(x, y) := \lim_{t \downarrow 0} \frac{\|x + ty\| - \|x\|}{t}, \quad x, y \in X. \tag{1.1.}$$

If X is a smooth normed linear space, by Theorem of James, $x \perp y$ if and only if $\tau(x, y) = 0$ i.e. $(y, x)_T = 0$.

Let now suppose that $x \mathrm{L} y$. It results $(\lambda y, x)_L = 0$ for all $\lambda \in K$. Since $(\lambda y, x)_T = 0$ for all $\lambda \in K$ (because we have by Lemma B, $(\lambda y, x)_T = \mathrm{Re}(\lambda y, x)_L$) we deduce that

$$\|x + t(\lambda y)\| \geqslant \|x\| \text{ for all } t \in \mathbb{R} \text{ and } \lambda \in K,$$

from where results $x \perp y$ in the sense of Birkhoff.

The lemma is proven.

Using the above lemma, we can prove the following result which gives a generalization for Theorem D.

1.2. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth normed linear space, $(\ ,\ )_L$ the L-semi-inner product which generates the norm $\|\cdot\|$, and E a closed linear subspace in X. If $x \in X$ and $x' \in E$, then the following conditions are equivalent:*

(i) *There exists $x'' \in E^{\mathrm{L}}$ such that*

$$x = x' + x''; \tag{1.2.}$$

(ii) $x' \in \mathscr{P}_E(x)$.

Proof. "(i) ⇒ (ii)". If $x'' := x - x' \in E^{\mathrm{L}}$, then by Lemma 1.1, we have $x'' \perp E$ in the sense of Birkhoff and by Lemma F we deduce that $x' \in \mathscr{P}_G(x)$.

"(ii) → (i)". If $x' \in \mathscr{P}_G(x)$ it results $x - x' \perp E$ i.e. there exists $x'' \in E^{\perp} = E^{\mathrm{L}}$ such that $x - x' = x''$ and relation (1.2) holds.

The theorem is proven.

1.3. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth normed linear space, $(\ ,\ )_L$ the L-semi-inner product which generates the norm and E a closed linear subspace in X. Then the following sentences are equivalent:*

(i) *for every $x \in X$ there exists $x' \in E$ and $x'' \in E^{\mathrm{L}}$ such that*

$$x = x' + x'';$$

(ii) *E is proximinal in X.*

The proof follows by the above theorem. We omit the details.

1.4. CONSEQUENCES. 1. Let $(X, \|\cdot\|)$ be a smooth normed linear space and E a linear subspace in X such that $S_E := \{g \in E \mid \|g\| \leqslant 1\}$ is weakly sequentially compact in X. Then for every $x \in X$ there exists $x' \in E$ and $x'' \in E^{\mathrm{L}}$ such that (1.2.) holds.

The proof follows by V. Klee's theorem [4] pp. 91 and by the above corollary.

2. If $(X, \|\cdot\|)$ is a smooth normed linear space and E is a finite-dimensional linear subspace in X, then for every $x \in X$, there exists $x' \in E$ and $x'' \in E^{\mathrm{L}}$ such that relation (1.2) holds.

The proof follows by Corollary 1.3, since every finite-dimensional linear subspace in X is proximinal.

3. Let $(X, \|\cdot\|)$ be a reflexive Banach space with a differentiable norm i.e. a smooth normed linear space, and E a closed linear subspace in X. Then for every $x \in X$ there exists $x' \in E$ and $x'' \in E^{\mathrm{L}}$ such that $x = x' + x''$ (see Theorem D.).

4. Let $(X, \|\cdot\|)$ be a normed linear space and suppose that X^* endowed with the canonical norm $\|\cdot\|$, $\|f\| := \sup_{\|x\|=1} |f(x)|$, is a smooth normed linear space. If Γ is a linear subspace in X^* and

(i) Γ is $\sigma(X^*, X)$-closed in X^*;

or

(ii) $S_\Gamma := \{g \in \Gamma \mid \|g\| \leqslant 1\}$ is compact in $\sigma(X^*, X)$;

or

(iii) S_Γ is weak * sequentially compact;

then for every $f \in X^*$ there exists $f' \in \Gamma$ and $f'' \in \Gamma^{\mathrm{L}}$ such that

$$f = f' + f''. \tag{1.3}$$

The proof results by Corollary 1.3 and by Corollary 2.5, Theorem 2.2 and Theorem 2.3 of [4] pp. 94–95. We omit the details.

Further, we shall establish the main result of this section what gives a necessary and sufficient condition of representation for the continuous linear functionals on smooth normed linear spaces in terms of best approximation.

1.5. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth normed linear space $(\ ,\ )_L$ the L-semi-inner product which generates the norm $\|\cdot\|$, $f \in X^* \setminus \{0\}$ and $g_0 \in \mathrm{Ker}(f)$, $x_0 \in X \setminus \mathrm{Ker}(f)$. Then the following sentences are equivalent:*

(i)
$$f(x) = \left(x, \frac{f(x_0)\,(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_L, \quad x \in X;$$

(ii)
$$\|f\| = \frac{|f(x_0)|}{\|x_0 - g_0\|};$$

(iii)
$$g_0 \in \mathscr{P}_{\mathrm{Ker}(f)}(x_0).$$

2.1. THEOREM. *Let* $(X, \|\cdot\|)$ *be a smooth normed linear space,* $(\ ,\ )_L$ *the* L-*semi-inner product which generates the norm,* G *a closed linear subspace in* X, $x_0 \in X \setminus G$ *and* $g_0 \in G$. *Then the following conditions are equivalent:*

(i) $g_0 \in \mathscr{P}_G(x_0)$;

(ii) *for every* $f \in (G \oplus [x_0])^*$ *such that* $G = \mathrm{Ker}(f)$, *we have*

$$f(x) = \left(x, \frac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_L, \ x \in G \oplus [x_0]; \tag{2.1}$$

(iii) *for every* $f \in (G \oplus [x_0])^*$ *such that* $G = \mathrm{Ker}(f)$, *we have*

$$\|f\| = \frac{|f(x_0)|}{\|x_0 - g_0\|}. \tag{2.2}$$

The proof follows by Theorem 1.5 for the smooth normed linear space $X_{x_0} = G \oplus [x_0]$. We omit the details.

Finally, we have:

2.2. COROLLARY. *Let* $(X, \|\cdot\|)$ *be a smooth normed linear space,* G *a closed linear subspace in* X. *Then the following sentences are equivalent:*

(i) G *is proximinal in* X;

(ii) *for every* $x_0 \in X \setminus G$ *and* $f \in (G \oplus [x])^*$ *such that* $\mathrm{Ker}(f) = G$, *there exists* $u_{x_0,f} \in G \oplus [x_0]$ *such that:*

$$f(x) = (x, u_{x_0,f})_L, \ x \in G \oplus [x]_0; \tag{2.3.}$$

(iii) *for every* $x_0 \in X \setminus G$ *and* $f \in (G \oplus [x_0])^*$ *such that* $\mathrm{Ker}(f) = G$, *there exists* $w_{x_0,f}, \in G \oplus [x_0]$ *such that*

$$|f(w_{x_0,f})| = \|f\|_{G \oplus [x_0]} \|w_{x_0,f}\|. \tag{2.4}$$

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