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THE BIFURCATION CURVE OF THE CHARACTERISTIC EQUATION PROVIDES THE BIFURCATION POINT OF THE NEUTRAL CURVE OF SOME ELASTIC STABILITY

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Abstract. The characteristic equation associated with the problem governing the buckling of a column has multiple solutions when the pair of the physical parameters belongs to some curve B. The neutral point corresponding to B proves to be a bifurcation point for the neutral curve of the problem.

The eigenvalue problem governing the buckling of a cantilever column in an elastic medium is expressed as [1]

$$\frac{d^4u}{dz^4}+2a^2\frac{d^2u}{dz^2}+b^4u=0\,;\quad z\in(0,l) \tag{1}$$

$$u(0)=\frac{du}{dz}(0)=\frac{d^2u}{dz^2}(l)=\frac{d^3u}{dz^3}(l)+2a^2\frac{du}{dz}(l)=0, \tag{2}$$

where $2a^2=PEI^{-1}$, $b^4=kEI^{-1}$, $E>0$ is the modulus of longitudinal elasticity, $I>0$ represents the minimum moment of inertia, l stands for the length of the column, $k=$ constant >0 is the rigidity coefficient of the elastic medium, the column being subjected to axial forces of compression $P>0$. The eigenfunction $u:[0,l]\to\mathbb{R}$, $u=u(z)$ belongs to $C^\infty[0,l]$; for b fixed a is the eigenvalue. Due to their physical significance the parameters a and b are real positive numbers. The characteristic equation corresponding to (1) is

$$\lambda^4+2a^2\lambda^2+b^4=0 \tag{3}$$

and it has four distinct solutions for $a\neq b$ (namely

$$\lambda_{1,2}=\pm\left(\sqrt{\frac{b^2-a^2}{2}}+i\sqrt{\frac{b^2+a^2}{2}}\right); \tag{4}$$

$$\lambda_{3,4}=\pm\left(\sqrt{\frac{b^2-a^2}{2}}-i\sqrt{\frac{b^2+a^2}{2}}\right)\ \text{if } a<b$$

$$(5)\qquad \lambda_{1,2} = \pm\, i\left(\sqrt{\frac{a^2-b^2}{2}} + \sqrt{\frac{a^2+b^2}{2}}\right);$$

$$\lambda_{3,4} = \pm\, i\left(\sqrt{\frac{a^2-b^2}{2}} - \sqrt{\frac{a^2+b^2}{2}}\right) \text{ if } a > b,$$

two distinct solutions $\lambda_1 = \lambda_4 = ia$, $\lambda_2 = \lambda_3 = -ia$ if $a = b > 0$ and a single trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ if $a = b = 0$. It follows that the curve (B) $a = b$, $a, b \geqslant 0$ in the (a, b) parameter space is the projection of the curve (in the four-dimensional space (Re λ, Im λ, a, b)) consisting of bifurcation points of the solution manifold M of (3). M is a surface with four distinct sheets for $a \neq b$; it has two distinct sheets if $a = b > 0$ and reaches the origin as $a = b = 0$. As (a, b) crosses B (for $a > 0$) two sheets coalesce while when $(a, b) \to (0, 0)$ all the four sheets approach the origin in the above-mentioned 4-dimensional space. When $a \nearrow b$ and the sheets $\lambda_1 = \lambda_1(a, b)$ and $\lambda_4 = \lambda_4(a, b)$ get closer and closer, the corresponding complex numbers λ_1 and $\lambda_4 = -\bar{\lambda}_1$ (the bar indicates the complex conjugation), becoming purely imaginary as B is crossed and the sheets coalescing into a curve which, for further increase of a (beyond B), splits into two other sheets $\lambda_1 = \lambda_1(a, b)$, $\lambda_4 = \lambda_4(a, b)$ corresponding to imaginary numbers λ_1 and $\lambda_4 \neq -\bar{\lambda}_1$. The sheets $\lambda_3 = \lambda_3(a, b)$ and $\lambda_4 = \lambda_4(a, b)$ are subjected to analogous transformation.

According to the position of the point (a, b) with respect to B, the secular equation corresponding to (1), (2) takes various forms. Since this equation describes the secular manifold S it follows that this manifold varies accordingly. Thus, if $(a, b) \notin B$, $a > b$, the secular equation, obtained introducing in (2) the general solution of (1)

$$(6)\qquad u = A_1 \cos \lambda_1 z + A_2 \sin \lambda_1 z + A_3 \cos \lambda_3 z + A_4 \sin \lambda_4 z$$

corresponding to (5), takes the form

$$(7)\qquad \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \lambda_1^2 \cos \lambda_1 l & \lambda_1^2 \sin \lambda_1 l & \lambda_3^2 \cos \lambda_3 l & \lambda_3^2 \sin \lambda_3 l \\ \lambda_1(\lambda_1^2 - 2a^2)\cdot \sin \lambda_1 l & -\lambda_1(\lambda_1^2 - 2a^2)\cdot \cos \lambda_1 l & \lambda_3(\lambda_3^2 - 2a^2)\cdot \sin \lambda_3 l & -\lambda_3(\lambda_3^2 - 2a^2)\cdot \cos \lambda_3 l \end{vmatrix} = 0$$

or, equivalently,

$$(7')\qquad 2b^4 + (b^4 - 2a^4 + a^2b^2)\cos 2\beta_1 + (b^4 - 2a^4 - a^2b^2)\cos 2\beta_2' = 0$$

where $\beta_1 = l\sqrt{\dfrac{a^2+b^2}{2}}$, $\beta_2' = \sqrt{\dfrac{a^2-b^2}{2}}\, l$. With the notation $y = a^2/b^2$ and $x = bl$, the equation (7′) writes also as

$$(7'')\qquad 4(1-y^2) + (4y^2 - 2y - 2)\sin^2 x\sqrt{\frac{1+y}{2}} + (4y^2 + 2y - 2)\sin^2 x\sqrt{\frac{y-1}{2}}$$

or, equivalently,

$$(7''')\qquad 2 - (2y^2 - y - 1)\cos(x\sqrt{2}\sqrt{y+1}) - (2y^2 + y - 1)\cos(x\sqrt{2}\sqrt{y-1}) = 0.$$

The secular manifold corresponding to (7) is a family of curves S_1 defined for $a > b$, $a, b > 0$. If $(a, b) \notin B$, $a < b$, the general solution of (1,) has also to the expression (6) where $\lambda_1 = \beta_2 + i\beta_1$, $\lambda_2 = -\lambda_1$, $\lambda_3 = \bar{\lambda}_1$ $\lambda_4 = -\bar{\lambda}_1$, $\beta_2 = \sqrt{\dfrac{b^2-a^2}{2}}\, l$, β_1 preserving its previous value. Therefore the corresponding secular equation may be obtained replacing in (7) β_2' by $i\beta_2$ and β_1 by β_1 and it has the form

$$(8)\qquad 2a^4 + (b^4 + 2a^4 - a^2b^2)\cos h^2\beta_2 + (b^4 - 2a^4 - a^2b^2)\cos^2 \beta_1 + 2a^4 = 0$$

or, equivalently,

$$(8')\qquad 2 - (2y^2 - y - 1)\cos(x\sqrt{2}\sqrt{y+1}) - (2y^2 + y - 1)\cosh(x\sqrt{2}\sqrt{1-y}) = 0.$$

This equation defines the family S_2 of secular curves for $a < b$, $a, b > 0$.

If $(a, b) \in B$, $a = b > 0$, hence $y = 1$, both the secular equations (7) and (8) hold. However, neither equation (7) nor equation (8) has $y = 1$ as a solution because these equations are not defined for $a = b$. In order to detect the secular points situated on B we must write the equation for the case $a = b$. So, for $a = b > 0$ the general solution of (1) has the form

$$(9)\qquad u = A_1 \cos az + A_2 \sin az + A_3 z \cos az + A_4 z \sin Az$$

such that the corresponding secular equation is expressed as

$$(10)\qquad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ -a^2 \cos al & -a^2 \sin al & -2a \sin al - a^2 l \cos al & 2a \cos al - a^2 l \sin al \\ -a^3 \sin al & +a^3 \cos al & -a^2 \cos al - a^3 l \sin al & -a^2 \sin al + a^3 l \cos al \end{vmatrix} = 0$$

or

$$(10')\qquad \cos^2 bl = \frac{b^2 l^2 - 1}{3},$$

or, equivalently,

$$(10'')\qquad \cos^2 x = \frac{x^2 - 1}{3}.$$

This equation has, for $x > 0$, a unique solution $x \simeq 1.1896$. Hence the only secular point (a, b) with $a = b > 0$ is $(a^*, a^*) = (1.1896/l;\ 1.1896/l)$ and the corresponding secular manifold is $S_3 = \{(a^*, a^*)\}$.

Finally let $a = b = 0$. Then the general solution of (1) is $u = A_1 + A_2 z + A_3 z^2 + A_4 z^3$ which introduced into boundary conditions (3) leads to $A_1 = A_2 = A_3 = A_4$ hence the point (0,0) is not a secular one.

Consequently, in the plane (x, y) the set of all secular points of (1), (2) is $S = S_1 \cup S_2 \cup S_3$ where S_1 and S_2 are two families of curves, each curve consisting of points (x, y) solutions of (7″) and (8′) respectively. S_3 consists of a single point $P^* = (x^*, y^*) = (1.1896\,; 1)$ which is a bifurcation point for S. Indeed, if (7‴) is written as $f(x, y) = 0$ and it is assumed for $a \geqslant b$ i.e. for $y \geqslant 1$ then simple algebra shows that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ at } P^*. \tag{11}$$

Because $P^* \in B$ there follows the conclusion expressed by the title of this note. However, (11) represents only a necessary condition for bifurcation; to be sure that P^* *is* a bifurcation point we must prove that the ratio $-(\partial f/\partial x)/(\partial f/\partial y)$ (which is a slope of a branch passing through P^*) has at least two distinct values as $(x, y) \to P^*$. This was shown numerically using a technique, special to the case when bifurcation is present, namely a continuation algorithm with P^* as a starting point [2]. The results of our computation are shown in Figure 1. If among all curves of S_1 and S_2 we retain only the closest to y—and x—axis respectively and attach the point P^* we obtain the neutral curve of problem (1),ı (2) [1].

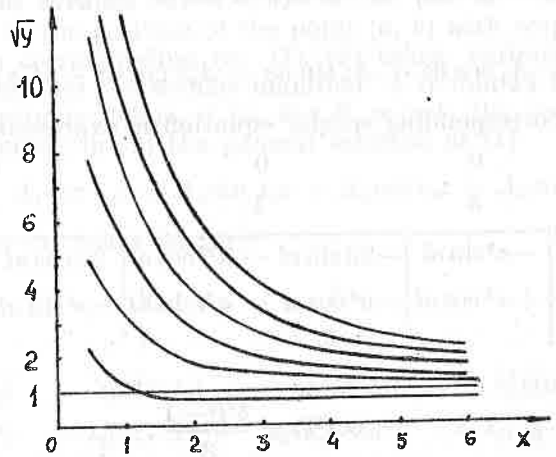

Fig. 1

The idea to connect bifurcation points (in the parameter space) for the characteristic and secular (and, consequently, neutral) equations came to us in trying to derive neutral manifolds for linear hydrodynamic and hydromagnetic stability problems involving high order ordinary differential equations and containing many parameters. In these cases usual numerical analysis failed since the secular equation has many forms in dependence of the position of the point P of the parameter space with respect to the bifurcation manifold B of the characteristic equation. This conferred a special role to the points $P^* \in M$ which belonged to the secular

manifold S because at these points the secular equation may be deduced from the secular equation corresponding to $P \notin B$ by appropriate differentiation followed by the limit $P^* \to P$. (In our case (10) may be obtained by differentiating (7) with respect to λ_3 and λ_4 and then letting $\lambda_3 \to \lambda_2$ and $\lambda_4 \to \lambda_1$ corresponding to the passage $(a, b) \to (a, a)$.) The occurrence of the Fréchet derivative suggested a connection with bifurcation of S and we conjectured : points $P^* \in B \cap S$, which, by definition, are bifurcation points for the characteristic equation (i.e. where this equation has multiple solutions), are also bifurcation points for the secular (and correspondingly, neutral) equation. (We recall that among the branches of the secular manifold, the neutral manifold realises the shortest distance to the axes). To etablish the nature of the bifurcation point P^* higher order derivatives of the secular equation must be used. The problem (1), (2) is the simplest possible since it involves only two parameters (a and b) and only fourth order equation. Moreover not all the branches of the secular equation corresponding to $a > b$ and $a < b$ emerge from P^* which seems to be the general case. Remark that equation (10) is a limit of (7) and P^* is a limit point of two among these branches (hence does not belong to them) which also is the general case. Further details will be published elsewhere [3].

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