MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 17, N° 2, 1988, pp. 147–152

ON THE DEGREE OF APPROXIMATION BY MODIFIED SZASZ OPERATOR

S. M. MAZHAR
(Kuwait)

1. Let

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n) \tag{1.1}$$

and

$$M_n(f, x) = (n+1) \sum_{k=0}^{n} P_{nk}(x) \int_0^1 P_{nk}(t) f(t)\, dt. \tag{1.2}$$

where

$$P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n.$$

and $f \in L[0,1]$. The operator (1.1) was introduced by Szasz [10] to approximate $f \in C[0, \infty)$ while operator (1.2) was introduced by Durrmeyer [6] who modified the well-known Bernstein's operator.

Concerning $f \in C^1 [0, a]$, $a > 0$ Stancu [9] proved that

$$\| S_n(f, x) - f \| \leq (a + \sqrt{a}) \frac{1}{\sqrt{n}} w\left(f^{(1)}, \frac{1}{\sqrt{n}}\right), \tag{1.3}$$

where $w(f^{(1)}, .)$ denotes the modulus of continuity of $f^{(1)}$. This estimate was sharpened by Varshney and Singh [11] by replacing a in (1.3) by $a/2$. Recently Singh [8] extended this result to $f \in C^{(r+1)}[0, a]$. His result is as follows:

THEOREM A. *Let $f \in C^{r+1}[0, a]$, $a>0$ and let $w(f^{(r+1)}, .)$ be its modulus of continuity. Then for $n \in N$, we have*

$$\| S_n^{(r)}(f, x) - f^{(r)} \| \leq \frac{r}{n} \| f^{(r+1)} \| + K_{n,r} \frac{1}{\sqrt{n}} w\left(f^{(r+1)}, \frac{1}{\sqrt{n}}\right),$$

where

$$K_{n,r} = \left[\frac{a}{2} + \frac{r}{2\sqrt{n}} + \frac{r^2}{4n} + \left(a + \frac{r^2}{4n}\right)^{\frac{1}{2}} \left(1 + \frac{r}{2\sqrt{n}}\right)\right].$$

Regarding $M_n(f)$, Derriennic [5] proved the following theorem.

THEOREM B. *If $f^{(r)}$ is continuous on* [0, 1], *then the sequence* $\left\{\frac{d^r}{dx^r} M_n(f, x)\right\}$ *converges uniformly to $f^{(r)}(x)$ and*

$$\sup_{x\in[0,1]}\left|\frac{(n+r+1)!(n-r)!}{n!(n+1)!}\frac{d^r}{dx^r}(M_n(f,x)) - f^{(r)}(x)\right| \leqslant K_1 w\left(f^{(r)}, \frac{1}{\sqrt{n}}\right),$$

where K_1 is a constant independent of f and n and $w(f^{(r)},.)$ denotes the modulus of continuity of $f^{(r)}$.

Recently the author and Totik [7] introduced modified Szász operator in the following manner.* Let.

$$(1.4)\qquad L_n(f, x) = n\,e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty e^{-nt}\frac{(nt)^k}{k!} f(t)\,dt,$$

where $f \in L\,[0, \infty)$. It is clear that $L_n(f, x)$ is a positive linear operator with $L_n(1, x) = 1$.

It may be remarked that earlier Butzer [2] defined another operator $W_u(f, x)$ which is as follows:

$$W_u(f, x) = u\,e^{-ux}\sum_0^\infty \frac{(ux)^k}{k!}\int_{\frac{k}{u}}^{\frac{k+1}{u}} f(t)\,dt,$$

where $f \in L[0, R]$ for every $R > 0$ and $f(x) = 0(x^k)$, as $x \to \infty$ for some $k > 0$.

Let $C_B[0, \infty)$ denote the class of bounded and uniformly continuous functions on $[0, \infty)$. We prove the following theorem.

THEOREM. *Let $f^{(r)} \in C_B[0, \infty)$ and let $f \in B[0, \infty)$, $r \geqslant 1$. If $w(f^{(r)}, \delta)$ denotes the modulus of continuity of $f^{(r)}$, then*

$$\sup_{x\in[0,\infty)} |L_n^{(r)}(f, x) - f^{(r)}(x)|\,\rho_n(x) \leqslant w\left(f^{(r)}, \frac{1}{\sqrt{n}}\right),$$

where

$$\rho_n(x) = \left\{1 + \sqrt{2x + \frac{(r+1)(r+2)}{n}}\right\}^{-1}.$$

To prove this theorem we need the following lemma ([1] p. 170).

LEMMA. *If $u \in L^p[0, \infty)$, $u^{(n)} \in L^r[0, \infty)$, p, $r \geqslant 1$, $n > 1$, then $u^{(k)} \in L^{(m)}[0, \infty)$ for $m \geqslant \max(p, r)$ and $k = 0,1,2, \ldots, n-1$.*

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* Later on the author came to know through Prof. D. D. Stancu that this operator was earlier introduced by Coatmelec [3] who proved a number of theorems concerning $L_n(f, x)$.

We use the special case $p = r = m = \infty$, where $u \in L^p[0,\infty)$, $p = \infty$ is interpreted as $\max_{t\in[0,\infty)} |u| \leqslant a < \infty$.

2. *Proof.* Since $\int e^{-u}\frac{u^r}{r!}\,du = -e^{-u}\sum_{\upsilon=0}^{r}\frac{u^\upsilon}{\upsilon!}$

we have

$$\int e^{-u}\left(\frac{u^{r+1}}{(r+1)!} - \frac{u^r}{r!}\right)du = -e^{-u}\frac{u^{r+1}}{(r+1)!}.$$

Then

$$L_n^{(1)}(f, x) = n(-n)e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty \frac{(nt)^k}{k!}e^{-nt}f(t)\,dt)$$

$$+\,n\,e^{-nx}\sum_1^\infty \frac{nk(nx)^{k-1}}{k!}\int_0^\infty \frac{(nt)^k}{k!}e^{-nt}f(t)\,dt$$

$$= -n^2e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty \frac{(nt)^k}{k!}e^{-nt}f(t)\,dt$$

$$+\,n^2e^{-nx}\sum_{k=0}^\infty \frac{(nx)^k}{k!}\int_0^\infty \frac{(nt)^{k+1}}{(k+1)!}e^{-nt}f(t)\,dt$$

$$= n^2e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty e^{-nt}\left\{\frac{(nt)^{k+1}}{(k+1)!} - \frac{(nt)^k}{k!}\right\}f(t)\,dt$$

$$= n^2e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty e^{-u}\left(\frac{u^{k+1}}{(k+1)!} - \frac{u^k}{k!}\right)f\left(\frac{u}{n}\right)\frac{du}{n}$$

$$= n\,e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\left\{\left[-e^{-u}\frac{u^{k+1}}{(k+1)!}f\left(\frac{u}{n}\right)\right]_0^\infty\right.$$

$$\left.-\int_0^\infty -e^{-u}\frac{u^{k+1}}{(k+1)!}f^1\left(\frac{u}{n}\right)\cdot\frac{1}{n}\,du\right\}$$

$$= e^{-nx}\sum_0^\infty \frac{(nx)^k}{k!}\int_0^\infty e^{-u}\frac{u^{k+1}}{(k+1)!}f^1\left(\frac{u}{n}\right)du.$$

By induction we get for $r \in N$

(2.1) $$L_n^{(r)}(f, x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!} f^{(r)}\left(\frac{u}{n}\right)du.$$

Since $\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\,du = 1$, we have

$$|L_n^{(r)}(f, x) - f^{(r)}(x)| = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left(f^{(r)}\left(\frac{u}{n}\right) - f^{(r)}(x)\right)du$$

$$\leqslant e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left|f^{(r)}\left(\frac{u}{n}\right) - f^{(r)}(x)\right|du$$

$$\leqslant w(f^{(r)}, \delta)e^{-nx}\sum_0^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left\{1 + \frac{\left|\frac{u}{n} - x\right|}{\delta}\right\}du$$

$$= w(f^{(r)}, \delta) + \frac{w(f^{(r)}, \delta)}{\delta}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left|\frac{u}{n} - x\right|dx.$$

$$= I_1 + I_2, \text{ say.}$$

Applying the Cauchy-Schwarz inequality,

(2.2) $$I_2 \leqslant \frac{w(f^{(r)}, \delta)}{\delta}\left\{\sum_{k=0}^{\infty} e^{-nx}\left[\frac{(nx)^k}{k!}\right]\left(\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left|\frac{u}{n} - x\right|du\right)^2\right\}^{\frac{1}{2}} \times$$

$$\times\left\{\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\right\}^{\frac{1}{2}} \leqslant \frac{w(f^{(r)}, \delta)}{\delta}\left\{\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\times\right.$$

$$\left.\times\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left(\frac{u}{n} - x\right)^2 du\right\}^{\frac{1}{2}} = \frac{w(f^{(r)}, \delta)}{\delta}\times$$

$$\times\left\{\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\left(x^2 - \frac{2x}{n}(k+r+1) + \frac{(k+r+1)(k+r+2)}{n^2}\right)\right\}^{\frac{1}{2}}$$

$$= \frac{w(f^{(r)}, \delta)}{\delta}\left\{x^2 - \frac{2x}{n}(r+1) + \frac{(nx + n^2x^2)}{n^2} - 2x^2 +\right.$$

$$\left. + \frac{(2r+3)x}{n} + \frac{(r+1)(r+2)}{n^2}\right\}^{\frac{1}{2}} = \frac{w(f^{(r)}, \delta)}{\delta}\left\{\frac{2x}{n} + \frac{(r+1)(r+2)}{n^2}\right\}^{\frac{1}{2}}$$

in view of the identities $e^{-nx}\sum_0^{\infty} k\frac{(nx)^k}{k!} = nx$ and

$$e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}k^2 = nx + n^2x^2.$$

Thus choosing $\delta = \frac{1}{\sqrt{n}}$, we have

$$|L_n^{(r)}(f, x) - f^{(r)}(x)| \leqslant w\left(f^{(r)}, \frac{1}{\sqrt{n}}\right)\left\{1 + \left(2x\frac{(r+1)(r+2)}{n}\right)^{\frac{1}{2}}\right\}$$

Writing $\rho_n(x) = \left\{1 + \sqrt{2x + \frac{(r+1)(r+2)}{n}}\right\}^{-1}$ we have

(2.3) $$\sup_{x\in[0,\infty)}|L_n^{(r)}(f, x) - f^{(r)}(x)|\rho_n(x) \leqslant w\left(f^{(r)}, \frac{1}{\sqrt{n}}\right).$$

This proves our theorem.

Note 1. From (2.1) it follows that $L_n^{(r)}(f, x)$ is a positive linear operator with $L_n^{(r)}(1, x) = 1$. Using Theorem 2.3 of R. A. Devore [3] we have

$$|L_n^{(r)}(f, x) - f^{(r)}(x)| \leqslant 2w(f^{(r)}, \alpha_n(x)), \quad \text{where}$$

$$\alpha_n^2(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-u}\frac{u^{k+r}}{(k+r)!}\left(\frac{u}{n} - x\right)^2 du$$

$$= \left\{\frac{2x}{n} + \frac{(r+1)(r+2)}{n^2}\right\} \text{ in view of (2.2).}$$

Thus

(2.4) $$|L_n^{(r)}(f, x) - f^{(r)}(x)| \leqslant 2w\left(f^{(r)}, \left\{\sqrt{2x + \frac{(r+1)(r+2)}{n}}\right\}\frac{1}{\sqrt{n}}\right)$$

$$\leqslant 2\left(1 + \sqrt{2x + \frac{(r+1)(r+2)}{n}}\right)w\left(f^{(r)}, \frac{1}{\sqrt{n}}\right).$$

This shows that our direct calculation gives sharper result in the sense that (2.4) contains an additional factor 2.

Note 2. Our theorem is an improvement and also an extension of the following theorem of Coatmelec [3].

THEOREM C. *If* $f \in C_B[0, \infty)$, *then*

$$|L_n(f, x) - f(x)| \leqslant 2(1 + \sqrt{2(x+1)})w\left(f, \frac{1}{\sqrt{n}}\right).$$

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Received 25.II.1988

Department of Mathematics
Kuwait University
Kuwait