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ON SOME CHARACTERISTICS OF GENERALIZED CONVEX SEQUENCES

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Let $a = (a_1, \ldots, a_n)$ be a real sequence. The operator L_{pq}, where p, q are real positive numbers, will be defined in the following way (see [1])

$$L_{pq}(a_k) = L_q(L_p(a_k)) = L_q(a_{k+1} - pa_k) = a_{k+2} - (p + q)a_{k+1} + pqa_k$$

If we have $p = q = 1$, then the operator L_{pq} reduces on the operator Δ^2. We shall say that a sequence $a = (a_n)$ is (p, q) — convex if inequality

$$L_{pq}(a_{k-2}) \geqslant 0$$

is valid, for all $k = 3, 4, \ldots, n$.

Let $a = (a_1, \ldots, a_n)$, $w = (w_1, \ldots, w_n)$ are two real sequences. Applying the generalized Abel transformation (see [2]) we shall say that following two identities hold (see [3])

$$(1)\qquad \sum_{i=1}^{n} w_i a_i = a_1 W_1 + \sum_{k=2}^{n} W_k L_p(a_{k-1}) \left(W_k = \sum_{j=k}^{n} p^{j-k} w_j \right)$$

$$(2)\qquad \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} a_i b_j = A_{11} a_1 b_1 + b_1 \sum_{r=2}^{n} A_{r1} L_p(a_{r-1}) + a_1 \sum_{s=2}^{m} A_{1s} L_q(b_{s-1}) + \sum_{r=2}^{n} \sum_{s=2}^{m} A_{rs} L_p(a_{r-1}) L_q(b_{s-1})$$

where

$$(3)\qquad A_{rs} = \sum_{i=r}^{n} \sum_{j=s}^{m} p^{i-r} q^{j-s} x_{ij} \quad (1 \leqslant r \leqslant n;\ 1 \leqslant s \leqslant m)$$

Now we will introduce the following notation

$$B_{r1}^{1} = \begin{cases} \displaystyle\sum_{i=r}^{n} \sum_{j=1}^{m} q^{j-1} \frac{p^{i-r+1} - t^{i-r+1}}{p - t} x_{ij}, & p \neq t \\ & r = 2, \ldots, n \\ \displaystyle\sum_{i=r}^{n} \sum_{j=1}^{m} (i - r + 1) p^{i-r} q^{j-1} x_{ij}, & p = t \end{cases}$$

$$
(4)\qquad B^2_{1s}=\begin{cases}\displaystyle\sum_{i=1}^{n}\sum_{j=s}^{m}p^{i-1}\frac{q^{j-s+1}-r^{j-s+1}}{q-r}x_{ij} & q\neq r\\ & s=2,\ldots,m\\ \displaystyle\sum_{i=1}^{n}\sum_{j=s}^{m}(j-s+1)q^{j-s}p^{i-1}x_{ij}, & q=r\end{cases}
$$

$$
B_{rs}=\begin{cases}\displaystyle\sum_{i=r}^{n}\sum_{j=s}^{m}\frac{p^{i-r+1}-t^{i-r+1}}{p-t}\cdot\frac{q^{j-s+1}-r^{j-s+1}}{q-r}x_{ij}, & p\neq t,\ q\neq r\\ & 2\leqslant r\leqslant n,\ 2\leqslant s\leqslant m\\ \displaystyle\sum_{i=r}^{n}\sum_{j=s}^{m}(i-r+1)(j-s+1)p^{i-r}q^{j-s}x_{ij}, & p=t,\ q=r\end{cases}
$$

Using identities (1) and (2), we can get the following identities :

$$
(5)\qquad \sum_{r=2}^{n}A_{r1}L_p(a_{r-1})=B^1_{21}L_p(a_1)+\sum_{r=3}^{n}B^1_{r1}L_{pt}(a_{r-2})
$$

$$
(6)\qquad \sum_{s=2}^{m}A_{1s}L_q(b_{s-1})=B^2_{12}L_q(b_1)+\sum_{s=3}^{m}B^2_{1s}L_{qr}(b_{s-2})
$$

$$
(7)\qquad \sum_{r=2}^{n}\sum_{s=2}^{m}A_{rs}L_p(a_{r-1})L_q(b_{s-1})=B_{22}\,L_p(a_1)L_q(b_1)+
$$

$$
+L_q(b_1)\sum_{r=3}^{n}B_{r2}L_{pt}(a_{r-2})+L_p(a_1)\sum_{s=3}^{m}B_{2s}L_{qr}(b_{s-2})+
$$

$$
+\sum_{r=3}^{n}\sum_{s=3}^{m}B_{rs}L_{pt}(a_{r-2})L_{qr}(b_{s-2}).
$$

By substitution of (5), (6) and (7) in (2) we obtain :

$$
(8)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_jx_{ij}=A_{11}\,a_1\,b_1+b_1L_p(a_1)B^1_{21}+a_1L_p(b_1)B^2_{12}+
$$

$$
+B_{22}L_p(a_1)L_q(b_1)+b_1\sum_{r=3}^{n}B^1_{r1}L_{pt}(a_{r-2})+a_1\sum_{s=3}^{m}B^2_{1s}L_{qr}(b_{s-2})+
$$

$$
+L_q(b_1)\sum_{r=3}^{n}B_{r2}L_{pt}(a_{r-2})+L_p(a_1)\sum_{s=3}^{n}B_{2s}L_{qr}(b_{s-2})+
$$

$$
+\sum_{r=3}^{n}\sum_{s=3}^{m}B_{rs}L_{pt}(a_{r-2})L_{qr}(b_{s-2})
$$

From identity (8) the following result follows directly :

Theorem 1. Let $x_{ij}(1\leqslant i\leqslant n;\ 1\leqslant j\leqslant m)$ be real numbers. Inequality :

$$
(9)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m}x_{ij}a_ib_j\geqslant 0
$$

is valid for $(p,\ t)$—convex sequence $a=(a_1,\ldots,a_n)$ and $(q,\ r)$-convex sequence $b=(b_1\ldots,b_n)$ if and only if :

$$
A_{11}=0,\quad B^1_{r1}=0\ (r=2,\ldots,n),\ B^2_{1s}=0\ (s=2,\ldots,m)
$$

$$
B_{r2}=0(r=2,\ldots,n),\ B_{2s}=0\ (s=3,\ldots,m)
$$

$$
B_{rs}\geqslant 0\ (r=3,\ldots,n;\ s=3,\ldots,m)
$$

Where A_{11}, B^1_{r1}, B^2_{1s} and B_{rs} are defined with (3) and (4).

Proof. The sufficient conditions follow directly from identity (8). We can check the necessity of the conditions using sequences contructed in theorem 1 of [5].

Remark 1. In special case, when $p=q$ and $r=t$, we obtain from theorem 1 the necessary and sufficient conditions for validity of (9) if we have two $(p,\ q)$—convex sequences.

Remark 2. The cases $p=q\to 1$ and $r=t\to 1$ are considered in [4].

Remark 3. The inequality (9) has been considered at first time by T. Popoviciu [6], but under different conditions.

Remark 4. For $q=p$, $r=t$, $b_j=p^{j-1}(j=1\ldots,m)$ and $\sum_{j=1}^{m}x_{ij}b_j=x_i$, $i=1\ldots,m$, from theorem 1 we obtain the result proved in [5]

Analogously, we can prove the following result :

Theorem 2. Let $x_{ij}(1\leqslant i\leqslant n;\ 1\leqslant j\leqslant m)$ be real numbers. Inequality (9) holds for $(p,\ t)$ — convex sequence $a=(a_1,\ldots,a_n)$ and q-monotonous sequence $b=(b_1,\ldots,b_n)$ if and only if :

$$
A_{1s}=0\quad (s=1,\ldots,m)
$$

$$
R_{r1}=0\ (r=2,3,\ldots,n),\ R_{2s}=0\ \ (s=2,\ldots,m)
$$

$$
R_{rs}\geqslant 0\ \ (r=3,\ldots,n;\ s=2,\ldots,m)
$$

where A_{1s} is defined by (3) and R_{rs} is defined by

$$
R_{rs}=\begin{cases}\displaystyle\sum_{i=r}^{n}\sum_{j=s}^{m}q^{j-s}\frac{p^{i-r+1}-t^{i-r+1}}{p-t}x_{ij}, & p\neq t\\ & 2\leqslant r\leqslant n,\ 1\leqslant s\leqslant m\\ \displaystyle\sum_{i=r}^{n}\sum_{j=s}^{m}(i-r+1)p^{i-r}q^{j-s}x_{ij}, & p=t\end{cases}
$$

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