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ON OSCILLATION CONDITIONS FOR LIÉNARD-TYPE EQUATION

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1. Introduction. In the present paper we deal with some oscillation conditions of a differential equation which is more general than the Liénard equation :

$$x'' + f(x)x' + g(x) = 0 \tag{1.1}$$

The equation (1.1) may be written as a first order system of differential equations :

$$\begin{cases} x' = y - F(x) \\ y' = -g(x) \end{cases} \tag{1.2}$$

In this paper, instead of (1.2) we consider the more general system, namely :

$$\begin{cases} x' = \varphi(x, y) \\ y' = -g(x), \end{cases} \tag{1.3}$$

and we suppose throughout this paper that the functions F, $g : \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are continuous, and conditions which guarantee the existence and uniqueness of the solution for each initial value problem for system (1.3) are satisfied. Moreover, we assume that the following conditions are satisfied :

$$\varphi \in C^1(\mathbb{R}^2), \quad \varphi(0,0) = 0, \quad xg(x) > 0, \quad x \neq 0, \tag{1.4}$$

and one can find a positive constant K as well as a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that :

$$K \leqslant \varphi_y'(x, y) \leqslant h(y), \quad (x,y) \in \mathbb{R}^2. \tag{1.5}$$

It is obvious that under conditions (1.4) and (1.5) the origin of the Euclidean plane is the unique critical point of system (1.3). Let us write $G(x) = \int_0^x g(s)\, ds$.

A solution $(x, y) : [t_0, \bar{t}) \to \mathbb{R}^2, (\bar{t} \leqslant +\infty)$, of system (1.3) is said to be *oscillatory* if there exist two sequences $\{t_n\}$ and $\{t'_n\}$, $t_n < t'_n < t_{n+1}$ tending monotonically to $\bar{t}$ such that $x(t_n) = y(t'_n) = 0$.

In [2] Gabrielle Villari proves that if

$$F(x) \geqslant -c > -\infty \quad \text{for } x > 0,$$

$$F(x) \leqslant c < +\infty \quad \text{for } x < 0,$$

and $xF(x) < 0$ for $|x| < \varepsilon > 0$, then all solutions of system (1.2) oscillate if and only if :

$$\limsup_{x\to+\infty} [G(x) + F(x)] = +\infty,$$

$$\limsup_{x\to-\infty} [G(x) - F(x)] = +\infty.$$

Later, Villari improves his own above stated result to the case : if

$$\limsup_{x\to+\infty} F(x) > -\infty,$$

$$\liminf_{x\to-\infty} F(x) < +\infty,$$

and $xF(x) < 0$ for $|x| < \varepsilon > 0$, then all solutions of system (1.2) oscillate if and only if :

$$\limsup_{x\to+\infty} [\Gamma_-(x) + F(x)] = +\infty$$

$$\limsup_{x\to-\infty} [\Gamma_+(x) - F(x)] = +\infty,$$

where

$$\Gamma_\pm(x) = \int_0^x \frac{g(s)}{1 + F_\pm(s)}\, ds,$$

$$F_\pm(x) = \max\{0, \mp F(x)\}.$$

This result was communicated by Gabrielle Villari at the XIth International Conference on Nonlinear Oscillations held at Budapest, August, 1987, as prof. A. Halanay kindly informed us.

It is easy to see that the above hypothesis is not fulfilled if we consider the equation :

$$x'' + bx' + cx = 0,$$

where b and c are real constants.

The aim of the present paper is to extend the result of G. Villari as well as some results of T. Hara and T. Yoneyama, [1]. We will study also system (1.3) under the following hypothesis :

$$\lim_{x\to\pm\infty} \varphi(x, 0) = \pm\infty, \tag{1.6}$$

which, for system (1.2) it means that :

$$\lim_{x\to\pm\infty} F(x) = \mp\infty.$$

In Section 2 we study the behaviour of the trajectory of system (1.3), which starts at a point of the *characteristic curve* $\varphi(x, y) = 0$ at $t=0$, and satisfies some repulsivity conditions. Our results are in connections with those of T. Hara and T. Yoneyama, [1].

In Section 3 we give a necessary and sufficient condition for oscillation of the solutions of system (1.3) when :

$$\liminf_{x\to+\infty} \varphi(x, 0) < +\infty$$

$$\limsup_{x\to-\infty} \varphi(x, 0) > -\infty,$$

hence we obtain an extension of the result of Gabrielle Villari.

In Section 4 we introduce sufficient conditions for intersection and non-intersection of the trajectory of system (1.3) with the characteristic curve, if (1.6) holds.

Throughout the paper we use no Liapunov function.

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2. Local behaviour at the origin. Let us write $D_1 = \{(x, y) \mid x > 0;$ $\varphi(x, y) > 0\}$ and $D_2 = \{(x, y) \mid x > 0, \varphi(x, y) < 0\}$.

LEMMA 2.1. *Every trajectory of system (1.3) passing through a point $B(x_0, y_0)$ $(x_0 \neq 0)$ which belongs to the characteristic curve intersects the y-axis at two points: $A(0, y_A)$ $(y_A \geqslant 0)$ and $C(0, y_C)$, $(y_C \leqslant 0)$. More precisely, if $x_0 > 0$ the solution of (1.3) leaving the point B at $t = 0$ either traverses the positive y-axis at some finite time $-t_A > 0$ as t decreases or tends to the origin as $t \to \underline{t}$ $(\underline{t} \geqslant -\infty)$ remaining in the region D_1, and traverses the negative y-axis at some finite time $t_C > 0$ as t increases or tends to origin as $t \to \bar{t}$ remaining in D_2.*

Proof. Let us suppose that $x_0 > 0$ and let $t = 0$ the moment at which the trajectory meets the characteristic curve at the point $B(x_0, y_0)$, $(\varphi(x_0, y_0) = 0)$. First we consider the case when $t \geqslant 0$. Since :

$$\frac{d}{dt}\varphi(x(t), y(t))\Big|_{t=0} = -\varphi'_y(x_0, y_0)\, g(x_0) < 0,$$

the solution $(x(t), y(t))$ enters in the region D_2 and does not intersect any more the characteristic curve as long $x(t) > 0$. Then $x' \leqslant 0$, $y' \leqslant 0$. Let us suppose that this trajectory starting in B does not meet the y-axis. Then we can find $\bar{x} \in [0, x_0)$ such that $x(t) \to \bar{x}$ and $y(t) \to -\infty$ for $t \to \bar{t}$ or $x(t) \to 0$, $y(t) \to 0$ for $t \to \bar{t}$, and $(x(t), y(t)) \in D_2$ for $t \in [0, \bar{t})$. If $y(t) \to -\infty$ for $t \to \bar{t}$, then from (1.5) we have that :

$$K(y_0 - y) \leqslant \varphi(x, y_0) - \varphi(x, y)$$

from where :

$$\varphi(x(t),\ y(t)) \leqslant \varphi(x(t),\ y_0) + K(y(t) - y_0).$$

It results that:

$$\lim_{t\to\bar t} \varphi(x(t),\ y(t)) = -\infty.$$

But this means that:

$$\lim_{t\to\bar t} x(t) = \lim_{t\to\bar t}\left(x_0 + \int_0^t \varphi(x(s),\ y(s))ds\right) = -\infty,$$

but that does not agree with $x(t) \geqslant 0$, $t \in [0, \bar t)$.

It results that the trajectory of system (1.3) passing through the point B either approaches the origin of the Euclidian plane for $t \to \bar t$ or it has to cross the y-axis at a finite distance, let's say $C(0, y_C)$ $(y_C < 0)$.

In a similar way, if we consider $t < 0$ we get the trajectory of system (1.3) passing through the point B either approaches the origin of the Euclidean plane for $t \to t$ or it has to cross the y-axis at a finite distance, et's say $A(O,\ y_A)$ $(y_A > 0)$.

Similar results hold in the case $x_0 < 0$.

Consider now the following conditions:

(i) $x\varphi(x,\ 0) > 0$, $0 < x < \varepsilon$;

(ii) there is a sequence $\{x_n\}$, $x_n > 0$, tending to zero such that $\varphi(x_n,\ 0) = 0$;

(iii) there is a positive number a such that $|\varphi(x,\ 0)| \neq 0$, for $0 < x \leqslant a$, and:

$$\frac{1}{\varphi(x,\ 0)}\cdot\int_{0+}^{x} \frac{g(s)}{\varphi(s,\ 0)}\, ds \geqslant \alpha \geqslant \frac{1}{4K}, \quad x \in (0,\ a];$$

(i') $x\varphi(x,\ 0) > 0$, $-\varepsilon < x < 0$;

(ii') there is a sequence $\{x_n\}$, $x_n < 0$, tending to zero such that $\varphi(x_n,\ 0) = 0$

(iii') there is a positive number a such that $|\varphi(x,\ 0)| \neq 0$ for $-a \leqslant x < 0$ and:

$$\frac{1}{\varphi(x,\ 0)}\int_{0-}^{x} \frac{g(s)}{\varphi(s,\ 0)}\, ds \geqslant \alpha \geqslant \frac{1}{4K}, \quad x \in [-a,\ 0);$$

These conditions are similar to those of Hara and Yoneyama [1].

LEMMA 2.2. *If all previous conditions are satisfied and moreover one of the conditions (i), (ii) or (iii) is satisfied and if we consider the trajectory of system (1.3) passing through a point* $B(x_0,\ y_0) \in \{(x,\ y)|\ \varphi(x,\ y) = 0\}$,

$x_0 > 0$, *then there is a* $t_A > 0$ *such that the trajectory crosses the y-axis at* $y_A < 0$ *and there is a* $-t_C > 0$ *such that the trajectory crosses the y-axis at* $y_C > 0$. *(The situation* $(x(t),\ y(t)) \to 0$ *is excluded.).*

Proof. The case (i). If $x \geqslant 0$, then $\varphi(x,0) > 0$. If $y > 0$ then $\varphi(x,\ y) > 0$, so $x'(t) > 0$. It follows that $x(t) > x_0$ and $x(t)$ does not approach the origin. If $y < 0$, then $y' < 0$, so $y(t)$ is decreasing and $y(t)$ does not approach the origin again.

The case (ii). For $t > 0$ we have $\varphi(x(t),\ y(t)) < 0$. If $\lim_{t\to\bar t} x(t) = 0$ there is $\tilde t \in (0,\ \bar t)$ such that $\varphi(x(\tilde t),\ y(\tilde t)) < 0$ and $\varphi(x(\tilde t),\ 0) = 0$, so, since $y \mapsto \varphi(x,\ y)$ is increasing for any x it follows $y(\tilde t) < 0$. From here $y(t) \leqslant y(\tilde t) < 0$ for $t \geqslant \tilde t$, so it is impossible to have $y(t) \to 0$.

The case (iii). If we have $\varphi(x,\ 0) > 0$, $x \in (0,\ a]$, then $y_0 < 0$ and $y_A \leqslant y_0 < 0$. If we have $\varphi(x,\ 0) < 0$, $x \in (0,\ a]$, then from Lemma 2.1, we have $y_A \leqslant 0$. We suppose that $y_A = 0$ and we will get a contradiction.

Let's take $y_A = 0$ and it results that $y(t) > 0$, $t \in [0,\ \bar t)$. Then for any $\varepsilon > 0$ and $x \in [\varepsilon,\ x_0)$ we have:

$$y(x) - y(\varepsilon) = \int_{\varepsilon}^{x} \frac{-g(s)}{\varphi(s,\ y(s))}\, ds = \int_{\varepsilon}^{x} \frac{g(s)}{|\varphi(s,\ y(s))|}\, ds > -\int_{\varepsilon}^{x} \frac{g(s)}{\varphi(s,\ 0)}\, ds,$$

$$y(x) - y(\varepsilon) > -\varphi(x,\ 0)\int_{\varepsilon}^{x} \frac{g(s)}{\varphi(x,\ 0)\ \varphi(s,\ 0)}\, ds.$$

If $\varepsilon \to 0$, then:

$$y(x) \geqslant -\varphi(x,\ 0)\int_{0+}^{x} \frac{g(s)}{\varphi(x,\ 0)\ \varphi(s,\ 0)}\, ds \geqslant -\alpha\cdot\varphi(x,\ 0).$$

Taking into account that:

$$\varphi(s,y(s)) \geqslant \varphi(s,\ 0) - K\alpha\cdot\varphi(s,\ 0) = (1 - K\alpha)\ \varphi(s,\ 0)$$

we have that if $\alpha K \geqslant 1$ we get a contradiction. Hence $\alpha K < 1$. If we repeat step by step the above argument, then we get $y(x) \geqslant \frac{\alpha}{1 - \alpha K}\cdot|\varphi(x,\ 0)|$.

If $\frac{\alpha}{1 - \alpha K} \geqslant \frac{1}{K}$ we get again a contradiction, so we suppose that $\frac{\alpha}{1 - \alpha K} < \frac{1}{K}$. In this way we build up the monotone and bounded sequence $\alpha_{n+1} = \alpha/(1 - \alpha_n K)$, and if we take $L = \lim_{n\to\infty} \alpha_n$, then $L^2K - L + \alpha = 0$, which has no real root since $4K\alpha \geqslant 1$. From this it follows that $y_A < 0$.

In a similar way we get that $y_C > 0$.

LEMMA 2.3. *If one of the conditions (i′), (ii′) or (iii′) is satisfied and if we consider the trajectory of system* (1.3) *passing through a point* $B(x_0, y_0) \in \{(x, y) \mid \varphi(x, y) = 0\}$, $x_0 < 0$, *then there is a* $t_A > 0$ *such that the trajectory crosses the y-axis at* $y_A > 0$ *and there is* $-t_C > 0$ *such that the trajectory crosses the y-axis at* $y_C < 0$.

Proof. It is analogous.

3. Intersection property and oscillations. Let us write $\varphi_{\pm}(x, 0) = \max\{0, \pm\varphi(x, 0)\}$, and:

$$\Gamma_{\pm}(x) = \int_0^x \frac{g(s)}{1 + \varphi_{\pm}(s, 0)}\, ds.$$

LEMMA 3.1. *Under conditions* (1.4) *and* (1.5) *and*

$$\lim_{x\to+\infty} \inf \varphi(x, 0) < +\infty, \quad \lim_{x\to-\infty} \sup \varphi(x, 0) > -\infty \tag{3.1}$$

for each $(x_0, y_0) \in D_1$ *with* $x_0 > 0$ *the trajectory of system* (1.3) *which passes through* (x_0, y_0) *crosses the characteristic curve for an* $x > x_0$ *if and only if*:

$$\lim_{x\to\infty} \sup\, [\Gamma_+(x) - \varphi(x, 0)] = +\infty \tag{3.2}$$

For each (x_0, y_0) *which* $x_0 < 0$ *and* $\varphi(x_0, y_0) < 0$ *the trajectory of system* (1.3) *which passes through* (x_0, y_0) *crosses the characteristic curve at an* $x < x_0$ *if and only if*:

$$\lim_{x\to-\infty} \sup\, [\Gamma_-(x) + \varphi(x, 0)] = +\infty \tag{3.3}$$

Proof. We prove here only the first statement. The second one runs in a similar way.

Let us suppose that:

$$\lim_{x\to+\infty} \sup\, [\Gamma_+(x) - \varphi(x, 0)] < +\infty \tag{3.4}$$

From (3.1) it results that one can find a monotone sequence $\{x_n\}$ tending to $+\infty$ such that $\varphi(x_n, 0) \to C < +\infty$, for $n \to \infty$. From (3.4) it results that $\{\Gamma_+(x_n)\}$ is a bounded increasing sequence, hence:

$$\int_0^\infty \frac{g(s)\, ds}{1 + \varphi_+(s, 0)} = C_1 < +\infty \tag{3.5}$$

Now, from (3.4) it follows that $\lim_{x\to+\infty} \inf\{\varphi(x, 0)\} > -\infty$, that is, one can find $C_2 > 0$ such that $\varphi(x, 0) \geqslant -C_2$, for $x > 0$. Then there is a positive a such that $\varphi(x, C_2 + a) > \varphi(x, 0) + C_2 + 1$. We show now that each trajectory of system (1.3) which starts in $R_1 = \{(x, y) \mid x > 0, y > C_2 + C_1 + a\}$ lies in $R_2 = \{(x, y) \mid x > 0, \ y > C_2 + a\} \subset D_1$. To

see that, let us consider a solution of system (1.3) with $(x(0), y(0)) \in R_1$ and we suppose that there is a $\tau \in [0, \bar{t})$ such that:

$$y(\tau) = C_2 + a \quad \text{and} \quad y(t) > C_2 + a, \ t \in [0, \tau) \tag{3.6}$$

From the above relations and (3.5) we have:

$$x'(t) = \varphi(x(t), y(t)) \geqslant \varphi(x(t), C_2 + a) > \varphi(x, 0) + C_2 + 1 > 1 + \varphi_+(x, 0)$$

and

$$y(\tau) = y_0 - \int g(x(t))\, dt > C_2 + C_1 + a - \int_{x(0)}^{x(\tau)} \frac{g(s)}{1 + \varphi_+(s, 0)}\, ds \geqslant C_2 + a.$$

But this is in contradiction with (3.6), so (3.3) is necessarily.

Let us see now that if (3.2) holds, then the trajectory of system (1.3) intersects the characteristic curve.

If

$$\lim_{x\to+\infty} \inf \varphi(x, 0) = -\infty,$$

there is a monotone sequence $\{x_n\} = \{x(t_n)\}$, $x_n \to +\infty$ such that $\varphi(x_n, 0) \to -\infty$, for $n \to \infty$. Then, since $y' < 0$,

$$\lim_{n\to\infty} \varphi(x_n, y(t_n)) \leqslant \lim_{n\to\infty} \varphi(x_n, |y_0|) \leqslant$$

$$\leqslant \lim_{n\to\infty} \varphi(x_n, 0) + |y_0| \cdot \max\{h(y) \mid y \in [0, |y_0|]\},$$

and

$$\lim_{n\to\infty} \inf \varphi(x_n, y(t_n)) \leqslant \lim_{n\to\infty} \varphi(x_n, 0) = -\infty.$$

Hence the trajectory intersects the characteristic curve.

If

$$\lim_{x\to+\infty} \sup \Gamma_+(x) = +\infty$$

it follows that

$$\lim_{x\to+\infty} \Gamma_+(x) = +\infty$$

Let us take a point (x_0, y_0), $x_0 > 0$, $\varphi(x_0, y_0) > 0$ and let us suppose that the trajectory which passes through (x_0, y_0) does not meet the curve $\varphi(x, y) = 0$. It results that along this trajectory there holds $\varphi(x, y) > 0$. Then:

$$0 < \varphi(x(t), y(t)) \leqslant \varphi(x(t), y_0) \leqslant \varphi(x(t), |y_0|) \leqslant$$

$$\leqslant \varphi(x(t), 0) + |y_0| \cdot \max\{h(y) \mid y \in [0, |y_0|]\} \leqslant a(1 + \varphi_+(x(t), 0))$$

for a positive constant a. But

$$y(t) = y_0 - \int_{t_0}^{t} g(x(s))\,ds \leqslant y_0 + \frac{1}{-a}\int_{x_0}^{x(t)} \frac{g(s)}{1+\varphi_+(s,0)}\,ds \to -\infty,$$

for $t \to \bar{t}$.

Then, using (1.5) we have

$$\liminf_{t\to\bar{t}} \varphi(x(t), y(t)) \leqslant \liminf_{t\to\bar{t}} [\varphi(x(t), y_0) + K(y(t) - y_0)] = -\infty,$$

which does not agree with the condition that $\varphi(x, y) > 0$ along the trajectory.

Now we can state :

THEOREM 3.2. *Let us suppose that for $x > 0$ one of the conditions (i), (ii) or (iii) is satisfied, for $x < 0$ one of the conditions (i'), (ii') or (iii') is satisfied and (1.4), (1.5) and (3.1) hold. Then all solutions of (1.3) oscillate if and only if (3.2) and (3.3) take place.*

Proof. It results from Lemmas 2.2, 2.3 and 3.1.

Remark. For $\varphi(x, y) = y - F(x)$ we get a result of G. Villari [2] and Theorem 5.3 of T. Hara and T. Yoneyama [1].

4. A different class of conditions. In the next theorems we consider that the following conditions are satisfied :

(4.1) $$\lim_{x\to\pm\infty} \varphi(x, 0) = \pm\infty.$$

Then it is obvious that there is an $x^* > 0$ such that $\varphi(x, 0) > 0$ $\varphi(-x, 0) < 0$ for $x > x^*$.

THEOREM 4.1. *Under conditions (1.4), (1.5) and (4.1) if*

(4.2) $$\limsup_{x\to\infty} \frac{1}{\varphi(x,0)} \int_{x^*}^{x} \frac{g(s)\,ds}{\varphi(x,0)} = \alpha > \frac{1}{K},$$

then each trajectory of system (1.3) which passes through the point (x_0, y_0) with $x_0 > 0$, $\varphi(x_0, y_0) > 0$ intersects the curve $\varphi(x, y) = 0$ for an $x > x_0$.

Under conditions (1.4), (1.5) and (4.1) if

(4.3) $$\limsup_{x\to-\infty} \frac{1}{\varphi(x,0)} \int_{-x^*}^{x} \frac{g(s)}{\varphi(s,0)}\,ds = \alpha > \frac{1}{K},$$

then each trajectory of system (1.3) which passes through the point (x_0, y_0) with $x_0 < 0$, $\varphi(x_0, y_0) < 0$ crosses the curve $\varphi(x, y) = 0$ for an $x < x_0$.

Proof. We prove here only the first part of the theorem. The other part runs in a similar way.

We prove that the equation $\varphi(x(t), y(t)) = 0$ has at least a root. Let us suppose that $\varphi(x(t), y(t)) > 0$ for $t \in [0, \bar{t})$. Then $x'(t) > 0$, $y'(t) < 0$ for $t \in [0, \bar{t})$ and $\lim_{t\to\bar{t}} x(t) = +\infty$. We may suppose that $x_0 > x^*$.

For the first step we show that $y(t)$ becomes negative at some moment. Indeed, from (4.1) we have that there is an $x_1 > x_0$ large enough such that $\varphi(x, 0) \geqslant |y_0| \cdot \max\{h(y) \mid y \in [0, |y_0|]\}$, $x > x_1$ and then

$$y(t) \leqslant y_0 - \int_{x_0}^{x(t)} \frac{g(s)}{\varphi(s, y_0)}\,ds = y_0 - \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s, y_0)}\,ds - \int_{x_1}^{x(t)} \frac{g(s)}{\varphi(s, y_0)}\,ds \leqslant$$

$$\leqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s, y_0)}\,ds - \frac{1}{2}\int_{x_1}^{x(t)} \frac{g(s)}{\varphi(s, 0)}\,ds \to -\infty,$$

for $t \to \bar{t}$.

We may thus suppose that $y_0 \leqslant 0$. For $\varepsilon \in \left(0, \alpha - \frac{1}{K}\right]$ there is a τ such that :

$$\int_{x_0}^{x(\tau)} \frac{g(s)}{\varphi(s, 0)}\,ds \geqslant (\alpha - \varepsilon) \cdot \varphi(x(\tau), 0).$$

From (1.5) we have $\varphi(x, -(\alpha - \varepsilon)\cdot\varphi(x, 0)) \leqslant [1 - k(\alpha - \varepsilon)]\cdot\varphi(x, 0)$, from where

$$\varphi(x, (\tau), y(\tau)) \leqslant \varphi\left(x(\tau), y_0 - \int_{x_0}^{x(\tau)} \frac{g(s)}{\varphi(s, 0)}\,ds\right) \leqslant \varphi\left(x(\tau), -\int_{x_0}^{(x\tau)} \frac{g(s)}{\varphi(s, y_0)}\,ds \leqslant\right.$$

$$\leqslant \varphi(x(\tau), -(\alpha - \varepsilon)\,\varphi(x(\tau), 0)) \leqslant (1 - k(\alpha - \varepsilon))\,\varphi(x(\tau), 0) < 0,$$

which is in opposition with $(x(t), y(t)) > 0$.

THEOREM 4.2. *If the function satisfies (1.4), (1.5) and (4.1) and if*

(4.4) $$\liminf_{x\to+\infty} \frac{1}{\varphi(x,0)} \int_{x^*}^{x} \frac{g(s)}{\varphi(s,0)}\,ds = \beta > \frac{1}{4K},$$

then each trajectory of system (1.3) which passes through the point (x_0, y_0) $x_0 > 0$, $\varphi(x_0, y_0) > 0$ crosses the curve $\varphi(x, y) = 0$. If the function φ satisfies (1.4), (1.5) and (4.1) and if

(4.5) $$\liminf_{x\to-\infty} \frac{1}{\varphi(x,0)} \int_{-x^*}^{x} \frac{g(s)}{\varphi(s,0)}\,ds = \beta > \frac{1}{4K}$$

then each trajectory of system (1.3) which passes through the point (x_0, y_0) with $x_0 < 0$, $\varphi(x_0, y_0) < 0$ crosses the characteristic curve.

Proof. Here we prove only the first statement of the theorem. The proof of the second one runs in a similar manner. As in the proof of Theorem 4.1 we may suppose that $x_0 > x^*$ and $y_0 \leqslant 0$. We also suppose that $\beta \leqslant \frac{1}{K}$, since otherwise we apply Theorem 4.1.

From (4.4) we have that if we choose $\varepsilon_1 \in \left(0, \beta - \frac{1}{4K}\right]$, then there is an $x_1 > x_0$ such that for any $x > x_1$ there holds:

$$\int_{x_0}^{x} \frac{g(s)}{\varphi(s,0)}\, ds > (\beta - \varepsilon_1)\, \varphi(x, 0).$$

Let t_1 be such that $x(t_1) = x_1$ and for $t > t_1$ we have:

$$y(t) \leqslant y_0 - \int_{x_0}^{x(t)} \frac{g(s)}{\varphi(s,0)}\, ds \leqslant -(\beta - \varepsilon_1)\, \varphi(x(t), 0)$$

and

$$y(t) \leqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s,0)}\, ds - \int_{x_1}^{x(t)} \frac{g(s)\, ds}{\varphi(s, -(\beta - \varepsilon_1)\, \varphi(s,0))} \leqslant$$

$$\leqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)\, ds}{\varphi(s,0)} - \frac{1}{1 - k(\beta - \varepsilon_1)} \int_{x_1}^{x(t)} \frac{g(s)}{\varphi(s,0)}\, ds.$$

But, from (4.4) if $0 < \varepsilon_2 < \min\left\{\frac{\varepsilon_1}{2},\ \beta k(\beta - \varepsilon_1)\right\}$ it can be found an $x_2 > x_1$ such that for $x > x_2$ it holds that:

$$\int_{x_1}^{x} \frac{g(s)\, ds}{\varphi(s,0)} \geqslant (\beta - \varepsilon_2)\, \varphi(x, 0).$$

Let t_2 be such that $x(t_2) = x_2$, then for $t > t_2$ we have:

$$y(t) \leqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)\, ds}{\varphi(s,0)} - \int_{x_1}^{x(t)} \frac{g(s)}{\varphi(s,0)}\, ds \leqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s,0)}\, ds -$$

$$-\frac{1}{1 - k(\beta - \varepsilon_1)} \int_{x_1}^{x(t)} \frac{g(s)}{\varphi(s,0)}\, ds \leqslant y_0 - \frac{\beta - \varepsilon_2}{1 - k(\beta - \varepsilon_1)}\, \varphi(x, 0).$$

Let us write:

$$\beta_1 = \frac{\beta - \varepsilon_2}{1 - k(\beta - \varepsilon_1)} > \beta.$$

For $t > t_2$ we have:

$$\varphi(x(t), y(t)) \leqslant \varphi(x(t), y_0 - \beta_1 \varphi(x, 0)) \leqslant (1 - \beta_1 k)\, \varphi(x, 0).$$

If $\beta_1 \geqslant \frac{1}{k}$, then $\varphi(x(t), y(t)) \leqslant 0$ and the theorem is proved. Let us suppose that $\beta_1 < \frac{1}{k}$, then for $t > t_2$ we have:

$$y(t) = y_0 - \int_{x_0}^{x_2} \frac{g(s)\, ds}{\varphi(s, y(s))} - \int_{x_2}^{x(t)} \frac{g(s)\, ds}{\varphi(s, y(s))} \leqslant -\frac{1}{1 - \beta_1 k} \int_{x_2}^{x(t)} \frac{g(s)\, ds}{\varphi(s,0)}.$$

We choose $0 < \varepsilon_3 < \frac{\varepsilon_2}{2}$, an $x_3 > x_2$ and $t_3 : x(t_3) = x_3$ such that for any $t > t_3$ it results:

$$y(t) \leqslant \frac{\beta - \varepsilon_3}{1 - k\beta_1} \cdot \varphi(x(t), 0).$$

Let us denote

$$\beta_2 = \frac{\beta - \varepsilon_3}{1 - k\beta_1} > \beta_1,$$

and there follows

$$\varphi(x(t), y(t)) \leqslant \varphi(x(t), -\frac{\beta - \varepsilon_3}{1 - k\beta_1}\, \varphi(x(t), 0)) \leqslant (1 - \beta_2 k) \cdot \varphi(x(t), 0).$$

If $\beta_2 \geqslant \frac{1}{k}$, then $\varphi(x(t), y(t)) \leqslant 0$, and the theorem is proved. We assume that $\beta_2 < \frac{1}{k}$. If we go on as above we get two sequences: $\{\beta_n\}$ is increasing, while $\{\varepsilon_n\}$ is decreasing and tends to zero. If we find a term β_n such that $\beta_n \geqslant \frac{1}{k}$, then the theorem is proved. If not, the sequence $\{\beta_n\}$ is bounded and monotone, and if we denote by L its limit, it results that $L \leqslant 1/k$. But L satisfies the equation $kL^2 - L + \beta = 0$; this equation has no real root for $4\beta k > 1$. It means that there is n such that $\beta_n \geqslant 1/k$.

In the next theorem we use the following condition:

$$0 < \varphi_y'(x, y) < h, \quad (x, y) \in \mathbb{R}^2 \tag{4.6}$$

THEOREM 4.3. *We assume that the function* φ *satisfies* (1.4), (4.1) *and* (4.6) *and if*

$$\lim_{x \to +\infty} \sup \frac{1}{\varphi(x, 0)} \int_{x^*}^{x} \frac{g(s)}{\varphi(s, 0)}\, ds = \mu < \frac{1}{4h}, \tag{4.7}$$

then there is at least a trajectory of system (1.3) which passes through the point (x_0, y_0) *with* $x_0 > 0$, $\varphi(x_0, y_0) > 0$ *and does not meet the characteristic curve.*

Proof. Let $x_0 = x^*$. First we show that for any $x_1 > x_0$ there is $y_0 > 0$ such that the trajectory passing through the point (x_0, y_0) does not meet the x-axis before x_1. Indeed, for

$$y_0 > \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s, 0)} ds$$

if we would have a t_1 with $x(t_1) < x_1$ and $y(t_1) = 0$, then:

$$0 = y(t_1) \geqslant y_0 - \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s, 0)} ds > 0 \quad (y(t) > 0 \text{ for } t < t_1)$$

Hence, we choose $\varepsilon \in \left(0, \frac{1}{4h} - \mu\right]$ and then there is $x_1 > x_0$ such that for $x > x_1$ the following inequalities hold:

$$\int_{x_0}^{x} \frac{g(s)}{\varphi(s, 0)} ds < (\mu + \varepsilon)\, \varphi(x, 0)$$

$$y_0 > \int_{x_0}^{x_1} \frac{g(s)}{\varphi(s, 0)} ds.$$

We show that the trajectory of system (1.3) which passes through (x_0, y_0) does not intersect the curve $y = -\frac{1}{2h} \cdot \varphi(x, 0)$, remaining for all t above it, and $\varphi(x, y) > 0$ if $y > -\frac{1}{2h} \varphi(x, 0)$. Indeed:

$$\varphi(x, y) > \varphi\left(x, -\frac{1}{2h} \varphi(x, 0)\right) = \varphi(x, 0) + \varphi_y'(x, \xi) \cdot \left[-\frac{1}{2h} \varphi(x, 0)\right] \geqslant$$

$$\geqslant \varphi(x, 0) - \frac{1}{2} \cdot \varphi(x, 0) = \frac{1}{2} \cdot \varphi(x, 0)$$

Now we assume that the trajectory of system (1.3) passing through (x_0, y_0) crosses the curve $y = -\frac{1}{2h} \cdot \varphi(x, 0)$, and let τ be the moment of the first intersection. Then $x(\tau) > x_1$, and

$$y(\tau) + \frac{1}{2h} \varphi(x(\tau), 0) \geqslant y_0 - \int_{x_0}^{x(\tau)} \frac{g(s)\, ds}{\varphi\left(s, -\frac{1}{2h} \varphi(s, 0)\right)} + \frac{1}{2h} \varphi(x(\tau), 0) \geqslant$$

$$\geqslant y_0 - 2\int_{x_0}^{x(\tau)} \frac{g(s)}{\varphi(s, 0)} ds + \frac{1}{2h} \varphi(x(\tau), 0) \geqslant y_0 +$$

$$+ \left(\frac{1}{2h} - 2\mu - 2\varepsilon\right) \cdot \varphi(x(\tau), 0) \geqslant y_0 > 0.$$

But this is a contradiction. Hence, the trajectory of system (1.3) passing through (x_0, y_0) does not meet the curve $y = -\frac{1}{2h} \varphi(x, 0)$, and this fact implies that $\varphi(x, y) > 0$ along the trajectory.

The result in the case when $x \to -\infty$ is similar and we omit it.

We have immediately the following result:

THEOREM 4.4. *If the conditions of the Lemma 2.3 are satisfied and Theorem 4.1 or Theorem 4.2 holds, then all solutions of system (1.3) oscillate.*

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