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APPROXIMATION BY POSITIVE OPERATORS IN THE SPACE $\mathscr{C}^{(p)}([a, b])$

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Abstract. In this paper we establish some Korovkin-type theorems in the space $\mathscr{C}^{(p)}([a, b])$, for the identity operator and, more generally, for finitely defined operators. Several examples and applications are also presented.

Introduction. This paper is mainly devoted to studying those subspaces H of $\mathscr{C}^{(p)}([a, b])$, the space of all p times continuous differentiable real functions defined on an interval $[a, b]$ $(p \geq 0)$, which satisfy the following condition

$$(*)\quad \begin{cases} \textit{for every sequence } (T_n)_{n \in \mathbb{N}} \textit{ of linear positive operators on } \mathscr{C}^{(p)}([a, b]) \textit{ such that } \lim\limits_{n \to \infty} T_n(h) = h \textit{ in } \mathscr{C}^{(p)}([a, b]) \textit{ for every} \\ h \in H, \textit{ one also has } \lim\limits_{n\to\infty} T_n(f) = f \textit{ in } \mathscr{C}^{(p)}([a, b]) \textit{ for every} \\ f \in \mathscr{C}^{(p)}([a, b]). \end{cases}$$

Here for positive operator we mean an operator which leaves invariant the cone

$$K_p = \{f \in \mathscr{C}^{(p)}([a, b]) : f(a) \geq 0, f'(a) \geq 0, \ldots, f^{(p-1)}(a) \geq 0, f^{(p)} \geq 0\}.$$

The most important results in this direction, or in some other related to it, have been obtained by B. Brosowski ([6]), H. B. Knoop-P. Pottinger ([10], [17]), G. I. Kudryavcev ([11]), R. M. Min'kova ([13]), B. Sendov-V. Popov ([20]), S. Stadler ([21]), A. A. Vasil'čenko ([22]).

In the first part of the paper we completely characterize property (*) in terms of envelopes and Choquet boundary.

Moreover we also study a property similar to (*) by replacing the identity operator with more general operators which are called finitely defined operators, in analogy of similar operators studied in spaces of continuous functions.

Finally, in the same spirit of the previous sections, the convergence of sequences of positive linear forms on $\mathscr{C}^{(p)}([a, b])$ toward discrete-type positive linear forms, is also investigated.

Several examples and applications are presented.

1. **Preliminaries and definitions.** Let $p \geq 0$ and let $\mathscr{C}^{(p)}([a, b])$ be the space of all p times continuous differentiable real functions on the real interval $[a, b]$. Let $\|\cdot\|$ denote the sup-norm. We shall consider the space $\mathscr{C}^{(p)}([a, b])$ endowed with the norm (which is equivalent to the usual norm of $\mathscr{C}^{(p)}([a, b])$)

$$\|f\|_p = \sup\{|f(a)|, |f'(a)|, \dots, |f^{(p-1)}(a)|, \|f^{(p)}\|\}$$

and with the order with respect to which the positive cone is

$$K_p = \{f \in \mathscr{C}^{(p)}([a, b]) : f(a) \geq 0, f'(a) \geq 0, \dots, f^{(p-1)}(a) \geq 0, f^{(p)} \geq 0\} =$$
$$= \{f \in \mathscr{C}^{(p)}([a, b]) : f \geq 0, f' \geq 0, \dots, f^{(p-1)} \geq 0, f^{(p)} \geq 0\}.$$

The key idea of this paper is to represent the space $\mathscr{C}^{(p)}([a, b])$ as a space of continuous functions on a compact space.

In fact the space $\mathscr{C}^{(p)}([a, b])$, endowed with the norm $\|\cdot\|_p$ and the order above indicated, is an AM-space with unit the function $1_p(t) = \sum_{j=0}^{p} (t-a)^j/j!$, $t \in [a, b]$. By the Kakutani representation theorem ([19], Th. 7.4), $\mathscr{C}^{(p)}([a, b])$ is order isomorphic to the space $\mathscr{C}(X_p)$, where X_p is the weakly compact set of real valued lattice homomorphisms of norm 1 on $\mathscr{C}^{(p)}([a, b])$.

Thus $X_p = \{\mu_0, \mu_1, \dots, \mu_{p-1}\} \cup \{\mu_t : t \in [a, b]\}$ where for all $f \in \mathscr{C}^{(p)}([a, b])$, $t \in [a, b]$ and $j = 0, 1, \dots, p-1$, $\mu_j(f) = f^{(j)}(a)$ and $\mu_t(f) = f^{(p)}(t)$.

More simply, we identify X_p with the subset $\{\omega_0, \omega_1, \dots, \omega_{p-1}\} \cup [a, b]$ where $\omega_i \in \mathbb{R}$, $\omega_0 < \dots < \omega_{p-1} < a$, endowed with the usual topology.

For $p \geq 1$ the order isomorphism $J_p : \mathscr{C}^{(p)}([a, b]) \to \mathscr{C}(X_p)$ is defined by $J_p f(\omega_j) = f^{(j)}(a)$, $j = 0, 1, \dots, p-1$ and $J_p f(t) = f^{(p)}(t)$, $t \in [a, b]$.

Its inverse is the operator $\Lambda_p : \mathscr{C}(X_p) \to \mathscr{C}^{(p)}([a, b])$, defined by putting for all $g \in \mathscr{C}(X_p)$ and $s \in [a, b]$:

$$\Lambda_p g(s) = \sum_{k=0}^{p-1} \frac{g(\omega_k)}{k!}(s-a)^k + \frac{1}{(p-1)!}\int_a^s (s-t)^{p-1} g(t)\,\mathrm{d}t.$$

For $p = 0$, J_p and Λ_p coincide with the identity operator on $\mathscr{C}([a, b])$.

Let $T : \mathscr{C}^{(p)}([a, b]) \to \mathscr{C}^{(p)}([a, b])$ be a positive linear operator, i.e., $T(K_p) \subset K_p$. Let H be a linear subspace of $\mathscr{C}^{(p)}([a, b])$ which contains a function h_0 with $h_0^{(j)}(a) > 0$, $j = 0, 1, \dots, p-1$ and $h_0^{(p)}(t) > 0$, $t \in [a, b]$.

Let us define the T-Korovkin closure of H (with respect to positive linear operators) by $\mathrm{Kor}_T(H) = \{f \in \mathscr{C}^{(p)}([a, b]) : \|T_n f - Tf\|_p \to 0$ for every sequence of positive linear operators $T_n : \mathscr{C}^{(p)}([a, b]) \to \mathscr{C}^{(p)}([a, b])$ such that $\|T_n h - Th\|_p \to 0$ for all $h \in H\}$.

If T is the identity operator, the corresponding Korovkin closure of H will be denoted simply by $\mathrm{Kor}(H)$.

H is called a T-Korovkin subspace of $\mathscr{C}^{(p)}([a, b])$ if $\mathrm{Kor}_T(H) = \mathscr{C}^{(p)}([a, b])$. A set $S \subset \mathscr{C}^{(p)}([a, b])$ is called a T-Korovkin set if the linear subspace of $\mathscr{C}^{(p)}([a, b])$ spanned by S is a T-Korovkin subspace.

Let $P : \mathscr{C}(X_p) \to \mathscr{C}(X_p)$ be the linear positive operator defined by $P = J_p T \Lambda_p$ and put $V = J_p(H)$; V is a linear subspace of $\mathscr{C}(X_p)$ which contains the strictly positive function $J_p h_0$.

Considering the usual order and the sup-norm in $\mathscr{C}(X_p)$ we define analogously $\mathrm{Kor}_P(V)$ and $\mathrm{Kor}(V)$. Then we have

(1.1) $$\mathrm{Kor}_T(H) = \Lambda_p(\mathrm{Kor}_P(V))$$

(1.2) $$\mathrm{Kor}(H) = \Lambda_p(\mathrm{Kor}(V)).$$

Denote by $M_+(X_p)$ the set of all positive Radon measures on X_p. For every $x \in X_p$ let δ_x be the Dirac measure at x. In that follows we shall use the following result (see [12], Th. 1.1).

THEOREM 1.1. *The following statements are equivalent:*
a) *V is a P-Korovkin subspace of $\mathscr{C}(X_p)$;*
b) *$v = \delta_x \circ P$ for all $x \in X_p$ and $v \in M_+(X_p)$ for which $v = \delta_x \circ P$ on V.*

Finally, using the norm $\|\cdot\|_p$ we define similarly the Korovkin closure of H with respect to linear contractions on $\mathscr{C}^{(p)}([a, b])$ (see [4]). By applying Cor. 1 of [4, p. 167] and (1.2) we obtain that a linear subspace H of $\mathscr{C}^{(p)}([a, b])$ which contains 1_p is a Korovkin subspace with respect to linear contractions iff it is a Korovkin subspace with respect to positive linear operators.

2. **Korovkin-type theorems for the identity operator.** Characterizations of $\mathrm{Kor}(V)$ in terms of envelopes, quasi peak points and Choquet boundary are given in [4]. Using (1.2) and these characterizations it is not difficult to prove the following result.

THEOREM 2.1. *Let H be a linear subspace of $\mathscr{C}^{(p)}([a, b])$ which contains a function h_0 such that $h_0^{(j)} > 0$, $j = 0, 1, \dots, p-1$ and $h^{(p)}(t) > 0$, $t \in [a, b]$. The following statements (a), (b), (c) are equivalent:*

(a) *H is a Korovkin subspace in $\mathscr{C}^{(p)}([a, b])$.*

(b) *For all $f \in \mathscr{C}^{(p)}([a, b])$, $j \in \{0, 1, \dots, p-1\}$ and $t \in [a, b]$,*

$$f^{(j)}(a) = \sup\{h^{(j)}(a) : h \in H, f - h \in K_p\} = \inf\{h^{(j)}(a) : h \in H, h - f \in K_p\},$$

$$f^{(p)}(t) = \sup\{h^{(p)}(t) : h \in H, f - h \in K_p\} = \inf\{h^{(p)}(t) : h \in H, h - f \in K_p\}.$$

(c) (c_1) *If $t \in [a, b]$ and μ is a positive linear form on $\mathscr{C}^{(p)}([a, b])$ such that $\mu(h) = h^{(p)}(t)$ for all $h \in H$, then $\mu(f) = f^{(p)}(t)$ for all $f \in \mathscr{C}^{(p)}([a, b])$.*

(c_2) *For all $j \in \{0, 1, \dots, p-1\}$ and for all positive linear form μ on $\mathscr{C}^{(p)}([a, b])$ such that $\mu(h) = h^{(j)}(a)$ for every $h \in H$, it follows that $\mu(f) = f^{(j)}(a)$ for all $f \in \mathscr{C}^{(p)}([a, b])$.*

In fact, (c_2) is equivalent to

(c_3) *For all $j \in \{0, 1, \dots, p-1\}$ there is $h \in H$ such that $h^{(j)}(a) = 0$, $h^{(i)}(a) > 0$, $i \in \{0, 1, \dots, p-1\} \setminus \{j\}$ and $h^{(p)}(t) > 0$, $t \in [a, b]$.*

The proof is based on the characterization of the Choquet boundary of V in terms of quasi peak points given in [4], Th. 8, p. 176 and on the topological properties of X_p.

Remark. The equivalences (b) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (b) generalize Theorem 4 and Theorem 7 of [6].

COROLLARY 2.2. *Let S be a subset of $\mathscr{C}^{(p)}([a, b])$ which contains the function t^p. Then $\{1, t, \ldots, t^{p-1}\} \cup S$ is a Korovkin set in $\mathscr{C}^{(p)}([a, b])$ if and only if $S^{(p)} = \{f^{(p)} : f \in S\}$ is a Korovkin set in $\mathscr{C}([a, b])$.*

Example 2.3. *Let $h \in \mathscr{C}^{(p)}([a, b])$ be such that $h^{(p)}$ is strictly convex or strictly concave on $[a, b]$. Then $\{1, t, \ldots, t^{p+1}, h\}$ is a Korovkin set in $\mathscr{C}^{(p)}([a, b])$.*

Example 2.4. *Let $0 \leqslant a < b$, $\lambda > 0$, $\mu > 0$, $\lambda \neq \mu$. Then $\{1, t, \ldots t^p, t^{p+\lambda}, t^{p+\mu}\}$ is a Korovkin set in $\mathscr{C}^{(p)}([a, b])$.*

Example 2.5. *Let $a, b \in \mathbb{R}$, $a < b$. Then the sets*

$$\{1, t, t^2, \ldots, t^p, t^{p+1}, t^{p+2}\} \tag{2.1}$$

$$\{1, t, t^2, \ldots, t^p, e^t, e^{2t}\} \tag{2.2}$$

are Korovkin sets in $\mathscr{C}^{(p)}([a, b])$.

Remark. For $p = 0$, (2.1) reduces to the classical result of P.P. Korovkin concerning positive linear operators on $\mathscr{C}([a, b])$.

For $p = 1$ (2.1) is a consequence of Theorem 7 of [6].

The result contained in Example 2.3 can be generalized in a more general situation.

In fact, given $m \in \mathbb{N}$, let $\mathscr{K}^m = \{f \in \mathscr{C}([a, b]) : [t_0, t_1, \ldots, t_m; f] \geqslant 0$ for all $a \leqslant t_0 < \ldots < t_m \leqslant b\}$ ($[t_0, t_1, \ldots, t_m; f]$ is the divided difference of f).

Let $h \in \mathscr{C}^{(p)}([a, b]) \cap \mathscr{K}^{p+2}$ and let H be the linear subspace of $\mathscr{C}^{(p)}([a, b])$ spanned by $1, t, \ldots, t^{p+1}, h$. Let $\mathscr{T} = \{I \subset [a, b] : I$ is a maximal closed interval on which $h^{(p)}$ is linear$\}$ and let $V = J_p(H)$.

From [3], Proposition 7 we deduce that $\mathrm{Kor}(V) = \{f \in \mathscr{C}(X_p) : f$ is linear on each $I \in \mathscr{T}\}$.

Using this result and (1.2) we obtain

THEOREM 2.6. *$\mathrm{Kor}(H)$ is the set of all $f \in \mathscr{C}^{(p)}([a, b])$ such that $f^{(p)}$ is linear on each $I \in \mathscr{T}$.*

3. Determining subspaces for positive linear forms.

We recall that, given a positive linear form $\mu : \mathscr{C}^{(p)}([a, b]) \to \mathbb{R}$, a linear subspace H of $\mathscr{C}^{(p)}([a, b])$ is called a determining subspace for μ if for every equicontinuous sequence $(\mu_n)_{n \in \mathbb{N}}$ of positive linear forms on $\mathscr{C}^{(p)}([a, b])$ such that $\lim\limits_{n \to \infty} \mu_n(h) = \mu(h)$ for every $h \in H$, we also have $\lim\limits_{n \to \infty} \mu_n(f) = \mu(f)$ for every $f \in \mathscr{C}^{(p)}([a, b])$.

As indicated in [1], this property is equivalent to the uniqueness of the positive extension of μ on H (i.e. if v is a positive linear form on $\mathscr{C}^{(p)}([a, b])$ such that $v = \mu$ on H, then $v = \mu$).

For a characterization of determining subspaces see [1], [7].

Let n be an integer, $n \geqslant p - 1$.

Let us consider a function $h_n \in \mathscr{C}^{(p)}([a, b])$ such that:

(3.1) if $n \geqslant p$, then $[t_0, t_1, \ldots, t_{n+1}; h_n] > 0$ for all $a \leqslant t_0 < \ldots < t_{n+1} \leqslant b$;
(3.2) if $n = p - 1$, then $h_n^{(p)}(t) > 0$ for all $t \in [a, b]$.

Let H_n be the linear subspace of $\mathscr{C}([a, b])$ spanned by the functions $1, t, \ldots, t^n, h_n$. We shall prove that H_n is a determining subspace for some positive discrete linear forms.

THEOREM 3.1. *Under the above assumptions, let $A : \mathscr{C}([a, b]) \to \mathbb{R}$ be a continuous linear form such that $A(f) \geqslant 0$ for all $f \in K_{n+1}$ and $A(f) = 0$ for all $f \in H_n$. Then $A = 0$.*

Proof. Suppose first that $n \geqslant p$. It suffices to prove that $A(g) \geqslant 0$ for any $g \in \mathscr{C}^{(p)}([a, b]) \cap \mathscr{K}^{n+1}$ and to apply Theorem 1 of [18].

For every $m \in \mathbb{N}$ let $T_m f$ be the zero function if $m = -1$ and the Taylor polynomial of degree m for f and the point a, if $m \geqslant 0$.

a) If $g \in \mathscr{C}^{(n+1)}[(a, b)] \cap \mathscr{K}^{n+1}$, then $g^{(n+1)} \geqslant 0$. We have $g - T_n g \in K_{n+1}$ and hence $A(g) \geqslant A(T_n g) = 0$.

b) In the general case for every $j \geqslant 1$ let $b_j = \sum\limits_{k=0}^{j} \binom{j}{k} g\left(\frac{k}{j}\right) x^k (1 - x)^{j-k}$ be the j-th Bernstein polynomial of $g \in \mathscr{C}^{(p)}([a, b]) \cap \mathscr{K}^{n+1}$. Then $\| b_j - g \|_p \to 0$ and $b_j \in \mathscr{C}^{(n+1)}([a, b]) \cap \mathscr{K}^{n+1}$. From a) it follows that $A(b_j) \geqslant 0$, hence $A(g) = \lim\limits_{j \to \infty} A(b_j) \geqslant 0$.

Suppose now that $n = p - 1$. We have $A(f) \geqslant 0$ for all $f \in K_p$ and $A(f) = 0$ for all $f \in H_{p-1}$. Let $h = h_{p-1} - T_{p-1} h_{p-1}$. For $f \in \mathscr{C}^{(p)}([a, b])$ let us put

$$m_f = \min \{f^{(p)}(t) : t \in [a, b]\}, \quad M_f = \max \{f^{(p)}(t) : t \in [a, b]\}.$$

Then $0 < m_h \leqslant M_h$. Let $g \in \mathscr{C}^{(p)}([a, b])$ and let

$$f_1 = T_{p-1}\, g + \frac{m_g}{M_h} h, \quad f_2 = T_{p-1}\, g + \frac{M_g}{m_h} h.$$

Then $g - f_1 \in K_p$, $f_2 - g \in K_p$, $f_1, f_2 \in H_{p-1}$.

It follows that $0 = A(f_1) \leqslant A(g) \leqslant A(f_2) = 0$, i.e. $A(g) = 0$.

Let now $k \geqslant 1$ be an integer. Define $\mu : \mathscr{C}^{(p)}([a, b]) \to \mathbb{R}$ by putting, for every $f \in \mathscr{C}^{(p)}([a, b])$

$$\mu(f) = \sum_{i=0}^{p-1} a_i f^{(i)}(a) + \sum_{j=1}^{k} c_j f^{(p)}(t_j), \tag{3.3}$$

where $a_i \geqslant 0$, $c_j \geqslant 0$ and $a \leqslant t_1 < \ldots < t_k \leqslant b$ are given real numbers.

Let $\sigma(t) = 1$ if $t \in \{a, b\}$ and $\sigma(t) = 2$ if $t \in]a, b[$. Write $m = \sum\limits_{j=1}^{k} (\operatorname{sign} c_j)\, \sigma(t_j)$ and let $n \geqslant m + p - 1$.

For $f \in \mathscr{C}^{(p)}([a, b])$ and $t \geqslant 0$ let us consider the functional K of J. Peetre (see, for example, [9], p. 4):

$$K(t, f, \mathscr{C}^{(p)}([a, b]), \mathscr{C}^{(n+1)}([a, b])) = \inf \{\| f - g \|_p + t \| g^{(n+1)} \| : g \in \mathscr{C}^{(n+1)}([a, b])\}.$$

If $v \in (\mathscr{C}^{(p)}([a, b]))'$ then $v \in (\mathscr{C}^{(n+1)}([a, b]))'$; let $\| v \|_p$ and $\| v \|_{n+1}$ be the corresponding norms, clearly equal.

Let P_n be the linear subspace of $\mathscr{C}^{(p)}([a, b])$ spanned by $1, t, \ldots, t^n$.

THEOREM 3.2. *Let $L : \mathscr{C}^{(p)}([a, b]) \to \mathbb{R}$ be a positive (and hence continuous) linear form.*

a) If $L = \mu$ on P_n, then for all $f \in \mathscr{C}^{(p)}([a, b])$:

$$|L(f) - \mu(f)| \leqslant \tag{3.4}$$

$$\leqslant (\| L \|_p + \| \mu \|_p) K \left[\frac{|(L - \mu)(t^{n+1})|}{(\| L \|_p + \| \mu \|_p)\ (n+1)!}, f, \mathscr{C}^{(p)}([a, b]), \mathscr{C}^{(n+1)}([a, b]) \right].$$

b) If $L = \mu$ on H_n, then $L = \mu$ on $\mathscr{C}^{(p)}([a, b])$. Hence H_n is a determining subspace for every positive linear form of type (3.3).

Proof. a) Suppose that $L = \mu$ on P_n. Let $A = \mu - L$ and let $f \in K_{n+1}$. If $m > 0$, let us denote by q the Hermite polynomial of $f^{(p)}$ for t_j with multiplicities $(\operatorname{sign} c_j)\sigma(t_j)$, $j = 1, \ldots, k$; then degree$(q) \leqslant m - 1$.

If $m = 0$, let q be the zero function on $[a, b]$.

Suppose that $c_k > 0$ and $t_k = b$. Then $q \geqslant f^{(p)}$. Let ω be the polynomial (of degree $\leqslant n$) for which $\omega^{(i)}(a) = f^{(i)}(a)$, $i = 0, 1, \ldots, p-1$ and $\omega^{(p)} = q$. We have $\omega - f \in K_p$, hence $L(f) \leqslant L(\omega) = \mu(\omega) = \mu(f)$. It follows that $A(f) \geqslant 0$, hence A is K_{n+1}-positive and $A \circ \Lambda_{n+1}$ is a positive linear form on $\mathscr{C}(X_{n+1})$.

If $c_k = 0$ or $t_k < b$, then $q \leqslant f^{(p)}$. By arguing as above we infer that $-A \circ \Lambda_{n+1}$ is a positive linear form on $\mathscr{C}(X_{n+1})$.

We conclude that $\| A \circ \Lambda_{n+1} \| = | A \circ \Lambda_{n+1}(1) | = A(1_{n+1}) = \dfrac{1}{(n+1)!} A(t^{n+1})$.

Since $\| A \|_{n+1} = \| A \circ \Lambda_{n+1} \|$, it follows that

$$\| A \|_{n+1} = \frac{1}{(n+1)!} | A(t^{n+1}) |. \tag{3.5}$$

Let now $f \in \mathscr{C}^{(p)}([a, b])$, $g \in \mathscr{C}^{(n+1)}([a. b])$. We have

$$| A(g) | \leqslant | A(g - T_n g) | + | A(T_n g) | \leqslant \| A \|_{n+1} \| g - T_n g \|_{n+1} =$$

$$= \frac{1}{(n+1)!} | A(t^{n+1}) | \| g^{(n+1)} \|, \ | A(f) | \leqslant | A(f - g) | + | A(g) | \leqslant \| A \|_p \| f - g \|_p +$$

$$+ \frac{1}{(n+1)!} | A(t^{n+1}) | \| g^{(n+1)} \| \leqslant (\| L \|_p + \| \mu \|_p) \Bigg(\| f - g \|_p +$$

$$+ \frac{|(L - \mu)(t^{n+1})|}{(\| L \|_p + \| \mu \|_p)(n+1)!} \| g^{(n+1)} \| \Bigg).$$

Taking the infimum when $g \in \mathscr{C}^{(n+1)}([a, b])$, we have a).

b) Let $L = \mu$ on H_n. Then $L = \mu$ on P_n and the above proof shows that A is K_{n+1}-positive or $-A$ is K_{n+1}-positive. In both cases $A = 0$ (cf. Th. 3.1).

Remark. For $p = 0$ and $n = 1$, (3.5) has been proved in [8]. (3.4) is similar to some inequalities given in [8] and [9].

4. **Korovkin-type theorems for finitely defined operators on $\mathscr{C}^{(p)}([a, b])$.** Let $p \geqslant 0$ and $k \geqslant 1$ be fixed. Let us consider an operator $T : \mathscr{C}^{(p)}([a, b]) \to \mathscr{C}^{(p)}([a, b])$ defined by putting for every $f \in \mathscr{C}^{(p)}([a, b])$ and $s \in [a, b]$,

$$\begin{aligned} T(f)(s) = {} & \sum_{r=0}^{p-1} \left[\sum_{j=0}^{p-1} a_{jr} f^{(j)}(a) + \sum_{i=1}^{k} c_{ir} f^{(p)}(x_{ir}) \right] \frac{(s-a)^r}{r!} + \\ & + \frac{1}{(p-1)!} \int_a^s (s-t)^{p-1} \left[\sum_{j=0}^{p-1} b_j(t) f^{(j)}(a) + \sum_{i=1}^{k} g_i(t) f^{(p)}(\Phi_i(t)) \right] dt \end{aligned} \tag{4.1}$$

where $a_{jr} \geqslant 0$, $c_{ir} \geqslant 0$, $b_j, g_i \in \mathscr{C}([a, b])$, $b_j \geqslant 0$, $g_i \geqslant 0$, $\Phi_i : [a, b] \to [a, b]$ is a continuous function and $x_{ir} \in [a, b]$ for all $i = 1, \ldots, k$ and $j, r = 0, 1, \ldots, p-1$. $\left(\text{For } p = 0,\ T(f)(s) = \sum_{i=1}^{k} g_i(s) f(\Phi_i(s)) \right)$.

T is a positive linear operator and hence it is continuous with respect to $\| \cdot \|_p$. Let $P = J_p T \Lambda_p$. P is a positive, linear, finitely defined operator on $\mathscr{C}(X_p)$. For analogy, we shall call the operator T a finitely defined operator on $\mathscr{C}^{(p)}([a, b])$ of order k. (Definitions and general results concerning finitely defined operators are given in [2]. Countably defined operators are defined and studied in [7]).

THEOREM 4.1. *Let T be a finitely defined operator on $\mathscr{C}^{(p)}([a, b])$ of order k of the form* (4.1).

Put $m_1 = \max \left\{ \sum_{i=1}^{k} (\operatorname{sign} c_{ir})\ \sigma(x_{ir}) : r = 0, 1, \ldots, p-1 \right\}$,

$$m_2 = \max \left\{ \sum_{i=1}^{k} (\operatorname{sign} g_i(t)) \sigma(\Phi_i(t)) : t \in [a, b] \right\}, \quad m = \max \{m_1, m_2\}.$$

(then $0 \leqslant m \leqslant 2k$). Let $n = m + p - 1$ and let H_n be the linear subspace of $\mathscr{C}^{(p)}([a, b])$ described in Section 3. Then H_n is a T-Korovkin subspace of $\mathscr{C}^{(p)}([a, b])$.

Proof. Let $V_n = J_p(H_n)$. Let $x \in X_p$ and $v \in M_+(X_p)$ such that $v = \delta_x \circ P$ on V_n. It follows that $v \circ J_p = \delta_x \circ J_p \circ T$ on H_n. Denote $v \circ J_p = L$, $\delta_x \circ J_p \circ T = \mu$ and apply Theorem 3.2.b); it follows that $v \circ J_p = \delta_x \circ J_p \circ T$ on $\mathscr{C}^{(p)}([a, b])$, i.e., $v = \delta_x \circ P$ on $\mathscr{C}(X_p)$. Using Theorem 1.1 we obtain $\operatorname{Kor}_P(V_n) = \mathscr{C}(X_p)$ and now (1.1) implies $\operatorname{Kor}_T(H_n) = \mathscr{C}^{(p)}([a, b])$.

COROLLARY 4.2. *Fix $p \geqslant 0$ and let us consider for every $q=0,1,\ldots,p$ the linear positive operator $T_q : \mathscr{C}^{(p)}([a,b]) \to \mathscr{C}^{(p)}([a,b])$ defined by putting for every $f \in \mathscr{C}^{(p)}([a,b])$ and $s \in [a,b]$*

$$T_q(f)(s) = \sum_{r=0}^{q} \frac{f^{(r)}(a)}{r!}(s-a)^r.$$

Then

(1) $\{1, t, \ldots, t^p\}$ *is a T_q-Korovkin set of $\mathscr{C}^{(p)}([a,b])$ for every $q = 0,1,\ldots,p-1$;*

(2) $\{1, t, \ldots, t^{p+1}\}$ *is a T_p-Korovkin set of $\mathscr{C}^{(p)}([a,b])$;*

(3) $\{1, t, \ldots, t^{p+2}\}$ *is a $(I-T_q)$-Korovkin set of $\mathscr{C}^{(p)}([a,b])$ for every $q = 0, 1, \ldots, p-1$ (where I denotes the identity operator on $\mathscr{C}^{(p)}([a,b])$).*

Proof. In fact, for $q \leqslant p-1$, T_q is a finitely defined operator on $\mathscr{C}^{(p)}([a,b])$ of order 1, where $g_1 = 0$, $c_{1r} = 0$, $b_j = 0$ for all j, $r = 0,1,\ldots,p-1$ and $a_{00} = \ldots = a_{qq} = 1$ and $a_{jr} = 0$ otherwise. In this case $m = 0$ and $n = p-1$, so the conclusion follows by Th. 4.1 with $h_n = t^p$.

In the case $q = p$, T_p is also a finitely defined operator on $\mathscr{C}^{(p)}([a,b])$ of order 1, where $a_{jr} = \delta_{jr}$, $b_j = 0$ for every $j,r = 0,1,\ldots,p-1$, $c_{1r} = 0$, $g_1 = 1$, $\Phi_1 = a$. In this case $m = 1$, $n = p$ and so we have the result with $h_n = t^{p+1}$.

The last case can be similarly proved since $I - T_q$ is obtained from (4.1) with $k=1$, $c_{1r} = 0$, $b_j = 0$, $g_1 = 1$, $\Phi_1(t) = t$ for every $j, r = 0,1,\ldots,p-1$ and $t \in [a,b]$, $a_{q+1,q+1} = \ldots = a_{p-1,p-1} = 1$ and $a_{jr} = 0$ otherwise.

Example 4.3. Let $\mu > 0$ be a fixed real number. Let us define $L^\mu : \mathscr{C}([0,a]) \to \mathscr{C}([0,a])$ as follows: for $f \in \mathscr{C}([0,a])$ let

$$L^\mu f(x) = \frac{1}{x^\mu}\int_0^x f(t)t^{\mu-1}\,dt$$

if $x \in \,]0,a]$ and $L^\mu f(0) = f(0)$.

Let $p \geqslant 0$. For each $f \in \mathscr{C}^{(p)}([0,a])$ we have $L^\mu f \in \mathscr{C}^{(p)}([0,a])$ (see [15], 2.5.8). Let us consider the operators $L_1^\mu = L^\mu$, $L_i^\mu = L^\mu \circ L_{i-1}^\mu$, $i \geqslant 2$. Then L_i^μ are K_p-positive linear operators (see [15], 2.5.9). It is easy to verify that $\| L_i^\mu 1 - 1\|_p \to 0$ and $\| L_i^\mu t^j \|_p \to 0$ for $j = 1,\ldots,p$. By using (1) of Coroll. 4.2 with $q = 0$ we obtain

$$\| L_i^\mu f - f(0)\|_p \to 0 \text{ for all } f \in \mathscr{C}^{(p)}([0,a]).$$

For $\mu = 1$ and $p = 0$ this result is contained in [5], Theorem 1; see also [14], Th. 1 and [16], Th. 1. Moreover Folgerung 2.19 (iv) of [9] yields:

$$|L_i^\mu f(x) - f(0)| \leqslant 4\omega_2\left(f, x\left(\frac{\mu}{\mu+2}\right)^{i/2}\right) + 2\left(\frac{\mu(\mu+2)}{(\mu+1)^2}\right)^{i/2}\omega_1\left(f, x\left(\frac{\mu}{\mu+2}\right)^{i/2}\right) \tag{4.2}$$

for all $f \in \mathscr{C}([0,a])$, $x \in [0,a]$ and $i \geqslant 1$.

In the sequel let $p \geqslant 1$, $k \geqslant 1$ and let S be a subset of $\mathscr{C}^{(p)}([a,b])$ which contains the functions $1, t, \ldots, t^{p-1}$. Suppose that for all distinct points $t_0, t_1, \ldots, t_k \in [a,b]$ there is $f \in S$ with $f^{(p)}(t_0) \neq f^{(p)}(t_i)$, $i = 1,\ldots,k$. For $f \in \mathscr{C}^{(p)}([a,b])$ let us denote

$$[f]^m(t) = \sum_{j=0}^{p-1}\frac{(f^{(j)}(a))^m}{j!}(t-a)^j + \frac{1}{(p-1)!}\int_a^t (t-s)^{p-1}(f^{(p)}(s))^m ds.$$

Let $[S]^m = \{[f]^m : f \in S\}$. Let H_S^k be the linear subspace of $\mathscr{C}^{(p)}([a,b])$ spanned by $\{1_p\} \cup S \cup [S]^2 \cup \ldots \cup [S]^{2k}$.

THEOREM 4.4. *H_S^k is a T-Korovkin subspace of $\mathscr{C}^{(p)}([a,b])$ for every finitely defined operator T on $\mathscr{C}^{(p)}([a,b])$ of order k.*

Proof. Let T be a finitely defined operator on $\mathscr{C}^{(p)}([a,b])$ of order k of the form (4.1) and put $P = J_p T \Lambda_p$. $P : \mathscr{C}(X_p) \to \mathscr{C}(X_p)$ is a positive, linear, finitely defined operator of order $p + k$. More precisely, for all $F \in \mathscr{C}(X_p)$ we have $P(F) = \sum_{i=0}^{p-1+k} \psi_i(F \circ \lambda_i)$, where $\psi_i \in \mathscr{C}(X_p)$, $\psi_i \geqslant 0$, $\lambda_i(x) = \omega_i$, $i = 0,1,\ldots,p-1$, $x \in X_p$ and

$$\lambda_{p-1+i}(x) = \begin{cases} x_{ij}, & x = \omega_i, \quad j = 0,1,\ldots,p-1 \\ \Phi_i(x), & x \in [a,b] \end{cases}$$

for all $i = 1,\ldots,k$.

Let $V = J_p(H_S^k)$. Let $x \in X_p$, $v \in M_+(X_p)$, $v = \delta_x \circ P$ on V. Then $v = \sum_{i=0}^{p-1} a_i\delta_{\omega_i} + a\lambda$, where $a_i \geqslant 0$, $i = 0,1,\ldots,p-1$, $a \geqslant 0$ and $\lambda \in M_+([a,b])$.

Since $v = \delta_x \circ P$ on V, it follows that the equality

$$\sum_{i=0}^{p-1} a_i\delta_{\omega_i} + a\lambda = \sum_{i=0}^{p-1}\psi_i(x)\delta_{\omega_i} + \sum_{i=0}^{k}\psi_i(x)\delta_{\lambda_i(x)} \tag{4.3}$$

holds on V. Using appropriate functions from V (see [2], Cor. 2.5) we can prove that (4.3) holds on $\mathscr{C}(X_p)$. Applying Theorem 1.1 and (1.1) we conclude that H_S^k is a T-Korovkin subspace of $\mathscr{C}^{(p)}([a,b])$.

For $k = 1$ we obtain the following result.

COROLLARY 4.5. *Let S be a subset of $\mathscr{C}^{(p)}([a,b])$ containing the functions $1, t, \ldots, t^{p-1}$ and such that $S^{(p)}$ separates the points of $[a,b]$* (cf. Coroll. 2.2). *Then $\{1_p\} \cup S \cup [S]^2$ is a T-Korovkin set of $\mathscr{C}^{(p)}([a,b])$ for every finitely defined operator T of order 1, and, in particular, for the identity operator.*

Example 4.6. *With the same notation of* Th. 4.2, *let* $S = \{1, t, \ldots, t^{p-1}, t^{p+1}\}$. *Then* H_S^k *is the subspace of* $\mathscr{C}^{(p)}([a, b])$ *spanned by*

$$\{1, t, \ldots, t^{p+2k}\}.$$

If $S = \{1, t, \ldots, t^{p-1}, e^t\}$, *then* H_S^k *is the subspace of* $\mathscr{C}^{(p)}([a, b])$ *spanned by* $\{1, t, \ldots, t^p, e^t, \ldots, e^{2kt}\}$.

In both cases H_S^k *is a T-Korovkin subspace of* $\mathscr{C}^{(p)}([a, b])$ *for every finitely defined operator* T *on* $\mathscr{C}^{(p)}([a, b])$ *of order* k.

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