MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 18, N° 1, 1989, pp. 81–88

GENERALIZATION OF THE THEOREM OF A. O. GUELFOND ON APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS

FRANCISCO LUQUIN MARTÍNEZ
(Bilbao)

1. Introduction. The problem of uniform approximation of functions with a finite number of derivatives by polynomials with integral coefficients (see pp. 125–138, [1]), is more complicated than the problem of uniform approximation of continuous functions. It seems that in order to get a good approximation we need to impose a number of conditions on the function. The best results in this respect are due to A. O. Guelfond [2] and R. M. Trigub [5]. For the standard notions of the theory of approximation by polynomials with integral coefficients we refer to the book by Le Baron O. Ferguson [1].

In this work, we stablish a generalization to n-variables of a theorem of A. O. Guelfond (p. 54 [2]). To this end we give some inequalities of Markov-Bernstein type for polynomials of several variables with positive coefficients in x_i and $1 - x_i$, in the cube n-dimensional $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$, (see [3]).

The statement of the theorem is as follows :

THEOREM 1. *"Let $(x_1, \ldots, x_n) \equiv f(\mathbf{X})$ be a continuous function, with continuous s_i first derivatives with respect to each x_i, on n-dimensional cube $D_n : 0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$. Let*

$$\omega_i(\delta) = \sup_{\substack{|\mathbf{X}-\mathbf{X}^*|<\delta \\ \mathbf{X}, \mathbf{X}^* \in D_n}} \left| \frac{\partial^{s_i} f(\mathbf{X})}{\partial x_i^{s_i}} - \frac{\partial^{s_i} f(\mathbf{X}^*)}{\partial x_i^{s_i}} \right|$$

$i = 1, \ldots, n$ be the moduli of continuity of all the partial derivatives of order s_i with respect to x_i. If all the numbers

$$\frac{1}{k_1! \ldots k_n!} \frac{\partial^k f(u_1, \ldots, u_n)}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \equiv \frac{1}{k_1! \ldots k_n!} f^{(k)}_{k_1, \ldots, k_n}(u) \qquad \begin{matrix} 0 \leqslant k_i \leqslant s_i \\ k = k_1 + \ldots + k_n \end{matrix}$$

are integers, where each u_i is either one or zero, there exists m_{i_0}, such that, for all $m_i > m_{i_0}$, such that $m_i = O(m_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, n$, then there is, a polynomial with integral coefficients $Q(\mathbf{X})$, of degrees $\leqslant m_i$, with respect to each x_i, such that, in the n-dimensional cube $D_n : 0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$, *the following bound holds*

$$|f^{(k)}_{k_1,\ldots,k_n}(\mathbf{X}) - Q^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < C\left(\sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right) + \frac{1}{m_i}}{m_i^{s_i-k}}\right)$$

for all $0 \leqslant k_i \leqslant s_i$, *with* $k = k_1 + \ldots + k_n$, *where* C *does not depend on* m_i, $i = 1, \ldots, n$."

Theorem 1 first appeared in Guelfond [2] for $n = 1$. The $n = 2$ case was shown by G. A. Zirnova [6]. Although the techniques employed are much in the same spirit as the ones used in [2] and [6], the additional difficulties which arise for $n > 2$ require some technical adjustments which may be of some independent interest.

2. Proof of Theorem 1. The proof of this theorem being fairly complex, we have to premise the following lemmas.

LEMMA 1. *The following holds*

$$(1)\quad \left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| = \left|\frac{d^k}{dx^k}[x^s(1-x)^{n-s}]\right| < \frac{2^k(n-s)^{n-s}s^s}{(n-k)^{n-k}},$$

where we take $0^0 = 1$, *for all* $x \in [0, 1]$; *and*

$$(2)\quad \left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| < \frac{n^k}{(1-x)^k x^k} x^{n-s}(1-x)^s,\quad 0 < x < 1.$$

Proof. Applying Leibniz's formula, we get

$$\left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| = \sum_{v=v_0}^{v_1} \frac{k!(n-s)!s!}{v!(k-v)!(n-s-v)!(s-k+v)!} x^{n-s-v}(1-x)^{s-k+v}$$

$$v_0 = \max(0, k-s),\quad v_1 = \min(k, n-s)$$

On the other hand, the maximum of $x^{n-s-v}(1-x)^{s-k+v}$ is attained at

$$x = \frac{n-s-v}{n-k}$$

being equal to

$$\max_{0 \leqslant x \leqslant 1}[x^{n-s-v}(1-x)^{s-k+v}] = \frac{(n-s-v)^{n-s-v}(s-k+v)^{s-k+v}}{(n-k)^{n-k}}$$

and since

$$\frac{(n-s)!s!}{(n-s-v)!(s-k+v)!} < (n-s)^v s^{k-v},$$

we then get

$$\left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| < \sum_{v=v_0}^{v_1} \frac{k!(n-s)^v s^{k-v}(n-s-v)^{n-s-v}(s-k+v)^{s-k+v}}{v!(k-v)!(n-k)^{n-k}} <$$

$$< \sum_{v=v_0}^{v_1}\binom{k}{v}\frac{(n-s)^{n-s}s^s}{(n-k)^{n-k}} < \frac{2^k(n-s)^{n-s}s^s}{(n-k)^{n-k}},\quad \text{for all } x \in [0, 1].$$

With regard to the second inequality,

$$\left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| \leqslant$$

$$\leqslant \sum_{v=v_0}^{v_1}\frac{k!(n-s)!s!}{v!(k-v)!(n-s-v)!(s-k+v)} x^{n-s-v}(1-x)^{s-k+v} <$$

$$< \frac{1}{(1-x)^k x^k}\sum_{v=v_0}^{v_1}\frac{k!}{v!(k-v)!}(n-s)^v s^{k-v} x^{n-s}(1-x)^s <$$

$$< \frac{n^k}{(1-x)^k x^k} x^{n-s}(1-x)^s,\quad \text{for all } x \in (0, 1).$$

Remark. It is clear that of (1), we can write

$$(3)\quad \left|\frac{d^k}{dx^k}[x^{n-s}(1-x)^s]\right| = O(n^{k-s}),\ \text{for fixed } k \text{ and } s.$$

LEMMA 2. *Let*

$$(4)\quad P(\mathbf{X}) = \sum_{v_1=0}^{m_1}\ldots\sum_{v_n=0}^{m_n} a_{v_1\ldots v_n} x_1^{v_1}\ldots x_n^{v_n}(1-x_1)^{m_1-v_1}\ldots(1-x_n)^{m_n-v_n}$$

be a polynomial of degree total m, *and degrees* m_i *with respect to each* x_i *with positive coefficients* $a_{v_1\ldots v_n} \geqslant 0$, *such that in the* n-*dimensional cube* $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$ *satisfy the inequality* $P(\mathbf{X}) \leqslant 1$. *Then*

$$|P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < \frac{m_1^{k_1}\ldots m_n^{k_n}}{(1-x_1)^{k_1}\ldots(1-x_n)^{k_n}x_1^{k_1}\ldots x_n^{k_n}}$$

where $k = k_1 + \ldots + k_n$, $0 \leqslant k_i \leqslant m_i$, $0 < x_i < 1$, $i = 1, \ldots, n$.

Proof.

$$|P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| = \left|\sum_{v_1=0}^{m_1}\ldots\sum_{v_n=0}^{m_n} a_{v_1\ldots v_n}\frac{\partial^k[x_1^{v_1}\ldots x_n^{v_n}(1-x_1)^{m_1-v_1}\ldots(1-x_n)^{m_n-v_n}]}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| <$$

$$< \sum_{v_1=0}^{m_1}\ldots\sum_{v_n=0}^{m_n} a_{v_1\ldots v}\frac{m_1^{k_1}}{(1-x_1)^{k_1}x_1^{k_1}}\ldots\frac{m_n^{k_n}}{(1-x_n)^{k_n}x_n^{k_n}}\cdot$$

$$\cdot x_1^{v_1}\ldots x_n^{v_n}(1-x_1)^{m_1-v_1}\ldots(1-x_n)^{m_n-v_n}$$

the last inequality by (2). The theorem follows from this last expression by simply bearing in mind that

$$P(\mathbf{X}) \leqslant 1.$$

LEMMA 3. *Under the conditions of lemma 2, then*

$$|P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < \frac{2^k m_1^{m_1}(m_1+1)\ldots m_n^{m_n}(m_n+1)}{(m_1-k_1)^{m_1-k_1}\ldots(m_n-k_n)^{m_n-k_n}}$$

where $k = k_1 + \ldots + k_n$, $0 \leqslant k_i \leqslant m_i$, $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$.

Proof. It is easy to deduce that

$$(5) \qquad a_{v_1\ldots v_n} \leqslant \left(\frac{m_1}{v_1}\right)^{v_1}\left(\frac{m_1}{m_1-v_1}\right)^{m_1-v_1}\ldots\left(\frac{m_n}{v_n}\right)^{v_n}\left(\frac{m_n}{m_n-v_n}\right)^{m_n-v_n}$$

for $0 \leqslant v_i \leqslant m_i$, $i = 1, \ldots, n$; where if some $v_i = m_i$ or $v_i = 0$ we take

$$\left(\frac{m_i}{v_i}\right)^{v_i}\left(\frac{m_i}{m_i-v_i}\right)^{m_i-v_i} = 1.$$

Reasoning with similar method as in lemma 2 and applying (1) and (5) the lemma follows.

If in the last lemma we impose some restrictive conditions on the coefficients of the polynomial, the result can be improved, using similar reasoning. We prove the following technical result.

LEMMA 4. *Let* $P(\mathbf{X})$ *be the polynomial in lemma 2 with total degree* $m > m_0$, $m = O(m_i)$, $i = 1, \ldots, n$. *If there is an arbitrary but fixed n-tuple* $(s_1, \ldots, s_n)$, $s = s_1 + \ldots + s_n$, $s_0 = 1 + \min\limits_{1 \leqslant i \leqslant n} s_i$, *such that*

$$(6) \qquad a_{v_1\ldots v_n} \leqslant m^{s+n}, \quad s_i + 1 \leqslant v_i \leqslant m_i - s_i - 1, \quad i = 1, \ldots, n$$

and

$$(7) \qquad a_{v_1\ldots v_n} \leqslant m^{s_0}$$

for those coefficients that have at least one $s_i + 1 \leqslant v_i \leqslant m_i - s_i - 1$ *but not all; then in n-dimensional cube* $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$ *for* $k = 1, \ldots, s$

$$(8) \qquad |P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < Cm^k, \quad k = k_1 + \ldots + k_n, \quad 0 \leqslant k_i \leqslant s_i$$

where C *does not depend on* m.

Proof. Let us divide $|P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})|$ into three summands S_1, S_2 and S_3. In the first sum S_1 the index v_i runs over all $0 \leqslant v_i \leqslant s_i$ and $m_i - s_i \leqslant v_i \leqslant m_i$, for $i = 1, \ldots, n$. In the second sum S_2 the summands corresponding to the index $s_1 + 1 \leqslant v_1 \leqslant m_1 - s_1 - 1, \ldots, s_n + 1 \leqslant v_n \leqslant m_n - s_n - 1$; whereas in S_3 the sum is taken over all the other index. This means that in S_3 some subindices v_i vary like in S_1, whereas other v_j vary like S_2; in other words, there are some v_i such that either $0 \leqslant v_i \leqslant s_i$ or $m_i - s_i \leqslant v_i \leqslant m_i$ and other v_j such that $s_j + 1 \leqslant v_j \leqslant m_j - s_j - 1$.

Let us now proceed to the bounds to each of these summands.

Applying (5) for the coefficients and (1) for the partial derivatives, we get

$$(9) \qquad |S_1| \leqslant \sum_{v_1}\cdots\sum_{v_n}\frac{2^{k_1}m_1^{m_1}}{(m_1-k_1)^{m_1-k_1}}\cdots\frac{2^{k_n}m_n^{m_n}}{(m_n-k_n)^{m_n-k_n}} < C_1 m^k$$

For the summand S_2, the coefficients are bounded by (6) while for the partial derivatives we apply the bound (see [2])

$$(10) \qquad \sum_{v=s+1}^{m-s-1}\frac{d^k}{dx^k}|[x^{m-v}(1-x)^v]| = O(m^{k-s-1})$$

$$(11) \qquad |S_2| \leqslant m^{s+n}\prod_{i=1}^{n} 0(m_i^{k_i-s_i-1}) < C_2 m^k.$$

Finally, for S_3 we use the bound (7) for the coefficients, and the bounds (3) and (10) for the partial derivatives.

When bounding

$$(12) \qquad A = \sum_{v_1}\ldots\sum_{v_n}\left|\frac{\partial^k[x_1^{v_1}\ldots x_n^{v_n}(1-x_1)^{m_1-v_1}\ldots(1-x_n)^{m_n-v_n}]}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right|$$

we will find in each term, groups of n factors of two types

$$\sum_{v_i} O(m_i^{k_i-v_i}) \text{ and } O(m_j^{k_j-s_j-1})$$

with $0 \leqslant v_i \leqslant s_i$. Let us group these terms in the following way: all those with only one $O(m_i^{k_i-v_i})$, those with two, $O(m_i^{k_i-v_i})$ and $O(m_j^{k_j-v_j})$, and following in this way, up to those with $(n-1)$ factors of this first type. For the bound for (12) we proceed in the following way:

$$A < \sum_{i_1=1}^{n}\left[\sum_{v_{i_1}=0}^{s_{i_1}} O(m^{k-v_{i_1}-(s-s_{i_1})-(n-1)})\right] +$$

$$+ \sum_{i_1=1}^{n}\sum_{\substack{i_2=1 \\ i_1\neq i_2}}^{n}\left[\sum_{v_{i_1}=0}^{s_{i_1}}\sum_{v_{i_2}=0}^{s_{i_2}} O(m^{k-(v_{i_1}+v_{i_2})-(s-s_{i_1}-s_{i_2})-(n-2)})\right] + \ldots +$$

$$+ \sum_{i_1=1}^{n}\ldots\sum_{\substack{i_{n-1}=1 \\ i_p\neq i_q}}^{n}\left[\sum_{v_{i_1}=0}^{s_{i_1}}\ldots\sum_{v_{i_{n-1}}=0}^{s_{i_{n-1}}} O(m^{k-(v_{i_1}+\ldots+v_{i_{n-1}})-(s-s_{i_1}-\ldots-s_{i_{n-1}})-1})\right].$$

The biggest values are obtained when $v_{i_1} = \ldots v_{i_{n-1}} = 0$. Therefore, the order of this expression is least or equal than

$$(n-1)\sum_{i_1=1}^{n}\ldots\sum_{\substack{i_{n-1}=1 \\ i_p\neq i_q}}^{n} O(m^{k-(s-s_{i_1}-\ldots-s_{i_{n-1}})-1})$$

since it contains the largest exponents. But this last sum is of the following order

$$(n-1)n^{n-1}[O(m^{k-s_1-1}) + \ldots + O(m^{k-s_n-1})].$$

Then,

$$|S_3| \leqslant O(m^{k+s_0-s_1-1}) + \ldots + O(m^{k+s_0-s_n-1}) = C_3 m^k. \quad (13)$$

Thus from (9), (11) and (13), the conclusion readily follows.

Now we can finally approach the proof of theorem 1.

Applying a theorem by G. A. Zirnova (see [6]), there exist m_{i_0} such that for all $m_i > m_{i_0}$, $i = 1, \ldots, n$, we cand find a polynomial with real coefficients $P(\mathbf{X})$ of degrees $\leqslant m_i$, with respect to each variable x_i, $i = 1, \ldots, n$, such that in the n-dimensional cube $D_n: 0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$ the following inequalities hold:

$$|f^{(k)}_{k_1,\ldots,k_n}(\mathbf{X}) - P^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < C \sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right)}{m_i^{s_i-k_i}}, \quad 0 \leqslant k_i \leqslant s_i, \; k = k_1 + \ldots + k_n, \quad (14)$$

where C does not depend on m_i, $i = 1, \ldots, n$.

Let m be, $m = \sup(m_i)$, clearly $m = O(m_i)$ if $m_i = O(m_j)$, $i, j = 1, \ldots, n$.

We can write this polynomial in the following way [2]

$$P(\mathbf{X}) = \sum_{v_1=0}^{m_1} \cdots \sum_{v_n=0}^{m_n} a_{v_1\ldots v_n} x_1^{v_1} \ldots x_n^{v_n} (1-x_1)^{m_1-v_1} \ldots (1-x_n)^{m_n-v_n},$$

and we decompose the coefficients $a_{v_1\ldots v_n}$ in the way $a_{v_1\ldots v_n} = r_{v_1\ldots v_n} + \alpha_{v_1\ldots v_n}$, $r_{v_1\ldots v_n}$ being the integer that is nearest to $a_{v_1\ldots v_n}$ and $|\alpha_{v_1\ldots v_n}| \leqslant 1/2$.

With this new notation, we can express

$$P(\mathbf{X}) = Q(\mathbf{X}) + R(\mathbf{X}) \quad (15)$$

where $r_{v_1\ldots v_n}$ and $\alpha_{v_1\ldots v_n}$ are the coefficients of $Q(\mathbf{X})$ and $R(\mathbf{X})$, respectively.

Let us first bound the coefficients of $R^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})$.

We can get by induction on v:

$$|\alpha_{\tau_1\ldots\tau_n}| < C \sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right)}{m_i^{s_i-v}} \quad (16)$$

where $\tau_i = v_i$ or $m_i - v_i$ for every $i = 1, \ldots, n$ and all v_i satisfying the conditions: $0 \leqslant v_i \leqslant s_i$, $v = v_1 + \ldots + v_n$, $i = 1, \ldots, n$.

The proof of this bound runs exactly as in [6] for the case $n = 2$.

Let us divide $R^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})$ into three summands $S_1 + S_2 + S_3$, in the same way as it is exposed in lemma 4.

Then S_1 contains all those summands whose coefficients admit the bound (16), while the partial derivatives are bounded by virtue of (3).

Therefore

$$|S_1| < C \sum_{v_1=0}^{s_1} \cdots \sum_{v_n=0}^{s_n} \sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right)}{m_i^{s_i-v}} \prod_{i=1}^{n} m_i^{k_i-v_i} \leqslant C \sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right)}{m^{s_i-k}} \quad (17)$$

For the summand S_2, the coefficients are replaced by 1/2, while for the partial derivatives, we apply the bound (10), obtaining:

$$|S_2| = \prod_{i=1}^{n} O(m_i^{k_i-s_i-1}) = O(m^{k-s-1}). \quad (18)$$

Finally, for S_3 we use the bound 1/2 for the coefficients, and the bounds (3) and (10) for the partial derivatives; and we proceed with identical reasoning as in lemma 4, we get

$$|S_3| \leqslant O(m^{k-s_1-1}) + \ldots + O(m^{k-s_n-1}) = O\left(\sum_{i=1}^{n} \frac{1}{m^{s_i-k+1}}\right) \quad (19)$$

Substituting (15) in (14), and bearing in mind (17), (18) and (19), we get

$$|f^{(k)}_{k_1,\ldots,k_n}(\mathbf{X}) - Q^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| = O\left(\sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right) + \frac{1}{m_i}}{m_i^{s_i-k}}\right),$$

since

$$\frac{1}{m^{s-k+1}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m^{s-k+1}} = O\left(\sum_{i=1}^{n} \frac{1}{m_i^{s_i-k+1}}\right)$$

The theorem is now proved.

COROLLARY. "*If under the conditions of the theorem* $\frac{1}{m_i} = O\left(\omega_i\left(\frac{1}{m_i}\right)\right)$, *then, the following inequalities hold,*

$$|f^{(k)}_{k_1,\ldots,k_n}(\mathbf{X}) - Q^{(k)}_{k_1,\ldots,k_n}(\mathbf{X})| < C \sum_{i=1}^{n} \frac{\omega_i\left(\frac{1}{m_i}\right)}{m_i^{s-k}}$$

where the constant C *does not depend on* m_i, $i = 1, \ldots, n$".

When $n = 1$ Guelfond's theorem is obtained.

REFERENCES

1. Ferguson, Le Baron O., *Approximation by polynomials with integral coefficients*, American Mathematical Society, **17** (1980).
2. Guelfond, A. O. *On uniform approximation by polynomials with integral coefficients*, Uspehi Mat. Nauk (N.S.), **10** (1955), no. 1 (63), 41–65. (Russian).
3. Lorentz, G. G., *The degree of approximation by polynomials with positive coefficients*, Math. Annalen, **151** (1963), 239–251.
4. Luquin, F., *Some properties of two-dimensional Bernstein-Polynomials*, to appear in J. Approximation Theory.
5. Trigub, R. M., *Approximation of functions by polynomials with integral coefficients*, Izv. Akad. Nauk SSSR Ser. Mat., **26** (1962), 261–280. (Russian).
6. Жирнова, Г. А. *Влияние дифференциальных свойств функции многих переменных на оценку приближения многочленами с целыми коэффициентами*, СБОРНИК ТРУДОВ, Теория функций, функциональный анализ и их приложения, Выпуск **15** (2), МОПИ, 1972 год, стр, 141–148

Received 20 VII 1988

Universidad del País Vasco
Facultad de Ciencias
Departamento de Matemáticas
Apartado 644
Bilbao-España