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ON THE ARITHMETICAL FUNCTION $d_k(n)$

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I. Introduction. The arithmetical function $d(n)$ is well known, i.e. the number of positive divisors of n. The function $d(n)$ can be interpreted as the number of the representations of n in the form xy $(x, y \in \mathbb{N}^*)$ so the following generalization is natural ([2], [5], [7], [10]) : The number of distinct solutions of the equation $x_1x_2\ldots x_k = n$ (where $x_1, x_2, \ldots, x_k$ run through independently the set of natural numbers) will be denoted by $d_k(n)$. For $k = 2$ we reobtain the function $d(n)$, and for $k = 1$ one has $d_1(n) = 1$.

Our aim is to prove some interesting properties of the function $d_k(n)$. Some of these results are generalizations to known theorems for $d(n)$ ([1], [3], [4]) and had not been anticipated (as far as we know) in the literature.

II. Some simple properties

1. *For* $n \geq 1$ *we have* $(k \geq 2,$ *fixed)*

$$(1) \qquad d_k(n) = \sum_{i|n} d_{k-1}(i)$$

Proof. ([10]) For x_k given, with $x_k | n$, the equation $x_1x_2\ldots x_{k-1}x_k = n$ has a number of $d_{k-1}\left(\dfrac{n}{x_k}\right)$ solutions. If x_k run through all the positive divisors of n, then evidently $\dfrac{n}{x_k}$ give also the divisors of n, so that we have $d_k(n) = \sum_{x_k|n} d_{k-1}(x_k) = \sum_{i|n} d_{k-1}(i)$, as required.

(2) COROLLARY. *The function* $d_k(n)$ *is multiplicative*

Proof. Use induction with respect to k. For $k = 1$ we have $d_1(n) = 1$ which is multiplicative. Assume the multiplicativity of $d_{k-1}(n)$; then it is well known that the function $\sum_{i|n} d_{k-1}(i)$ is also multiplicative ([1], [3]) so relation (1) implies the desired property.

Remark. Another argument may be found e.g. in [2].

2. *For* $k \geq 1$, $\operatorname{Re} s > 1$, *one has*

(3) $$(\zeta(s))^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ *denotes Riemann's zeta function.*

Proof. ([3]) For $\operatorname{Re} s > 1$ the series of $\zeta(s)$ is absolutely convergent, and we can write : $(\zeta(s))^k = \sum_{\nu_1=1}^{\infty} \frac{1}{\nu_1^s} \cdots \sum_{\nu_k=1}^{\infty} \frac{1}{\nu_k^s}$

Now, the number of systems $(\nu_1, \nu_2, \ldots, \nu_k)$ with $\nu_1 \nu_2 \ldots \nu_k = n$ is exactly $d_k(n)$. Thus we have : $(\zeta(s))^k = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu_1 \nu_2 \ldots \nu_k = n} 1 = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$.

3. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ *be the canonical representation (i.e. the prime factorization) of* n. *Then we have* :

$$d_k(n) = \frac{(k+\alpha_1-1)!}{\alpha_1!(k-1)!} \cdots \frac{(k+\alpha_r-1)!}{\alpha_r!(k-1)!} = \binom{k+\alpha_1-1}{\alpha_1} \cdots \binom{k+\alpha_r-1}{\alpha_r}$$

(4)

Proof. First method. First we show that $d_k(p^\alpha) = \binom{k+\alpha-1}{\alpha}$ for a prime p and then the application of (2) yields equality (4). The number of distinct solutions of $p^\alpha = x_1 x_2 \ldots x_k$ can be reduced to a well-known combinatorial problem, see e.g. [9].

Second method ([7]) From Euler's formula we derive $(\zeta(s))^k = \prod_p (1 - p^{-s})^{-k} = \prod_p (1 + p^{-s} + p^{-2s} + \ldots)^k = \prod_p \left(1 + \frac{k}{p^s} + \ldots\right)$ and taking into account (3), relation (4) follows by coefficient identification.

Remark. For a third method see [2].

(5) **4.** *If* $m|n$, *then* $d_k(m) \leq d(n)$

(6) $$d_k(n) \leq d_{k-1}(n) d(n) \quad \textit{for } k \geq 2$$

(7) $$d_k(n) \leq (d(n))^{k-1} \quad \textit{for } k \geq 2$$

Proof. Let $n = \prod p_i^{\alpha_i}$, $m = \prod p_i^{\beta_i}$ be the prime factorizations of n and m, respectively, and suppose $\beta_i \leq \alpha_i$ (Here some of α_i, β_i can vanish). Writing $d_k(n)/d_k(m) = \prod_i [(k+\alpha_i-1)!/\alpha_i!] \cdot [\beta_i!/(k+\beta_i-1)!]$ and remarking that the terms of the fraction $\frac{k+\beta_i}{1+\beta_i} \cdots \frac{k+\beta_i+(\alpha_i-\beta_i-1)}{(\alpha_i-\beta_i)+\beta_i}$ are ≥ 1, we obtain (5). Relation (6) is a simple consequence of (1) and (5) : $d_k(n) \leq d_{k-1}(n) \sum_{i/n} 1 = d_{k-1}(n) \cdot d(n)$. (7) results by the successive application of (6) (using $d_1(n) = 1$).

(8) **5.** *One has* $d_k(n) = 0(n^\varepsilon)$ *for all* $\varepsilon > 0 (k \geq 2)$

Proof. Let $n = \prod_{i=1}^{r} p_i$ where $p_1 < p_2 < \ldots < p_r$ are primes. Then for $\theta > 0$ we have $\frac{d(n)}{n^\theta} \leq \frac{\alpha_1+1}{2^{\theta\alpha_1}} \cdot \frac{\alpha_2+1}{3^{\theta\alpha_2}} \cdots \frac{\alpha_r+1}{(r+1)^{\theta\alpha_r}}$, because of $p_1 \geq 2$, $p_2 \geq 3$, $p_r \geq r+1$. By the inequality $\alpha + 1 \leq 2^\alpha$, the terms in the right-hand are less than $1/\theta$. For $k > 2^{1/\theta}$, the expressions $\frac{\alpha_{k-1}+1}{k^{\alpha\theta_{k-1}}}$ in the above product are less than $\frac{\alpha_{k-1}+1}{2^{\alpha_{k-1}}} \leq 1$. Thus we can write :

$\frac{d(n)}{n^\theta} \leq (1/\theta)^{2^{1/\theta}} = A$. Let $\varepsilon' = 2\theta$. There folows : $\lim_{n\to\infty} \frac{d(n)}{n^{\varepsilon'}} \leq \lim_{n\to\infty} \frac{A}{n^\theta} = 0$. Using now (7), with the notation $\varepsilon = (k-1)\varepsilon'$ we get $\lim_{n\to\infty} \frac{d_k(n)}{n^\varepsilon} \leq \lim_{n\to\infty} \left(\frac{d(n)}{n^{\varepsilon'}}\right)^{k-1} = 0$. Since the selection of ε' and θ is always possible in function of ε, the proof is finished.

6. *We have the following inequalities* :

(9) $$k^{\omega(n)} \leq \prod_{i=1}^{r} \left(1 + \frac{k-1}{\alpha_i}\right)^{\alpha_i} \leq d_k(n) \leq k^{\Omega(n)}$$

where $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors of n and the number of all prime factors of n, respectively.

Proof. The obvious inequalities $\frac{p}{q} > \frac{p-1}{q-1} > \frac{p-2}{q-2} > \ldots > \frac{p-h+1}{q-h+1}$, $(p \geq q \geq h > 1)$ imply at once the relations $(p/q)^h \leq \binom{p}{h} \Big/ \binom{q}{h} \leq \left(\frac{p-h+1}{q-h+1}\right)^h$. Apply this double-inequality for $q = h$ in order to obtain :

$(p/h)^h \leq \binom{p}{h} \leq (p-h+1)^h$. Selecting $p = \alpha + k - 1$, $h = \alpha$, we get

(10) $$\left(1 + \frac{k-1}{\alpha}\right)^\alpha \leq \binom{\alpha+k-1}{\alpha} \leq k^\alpha$$

Using (4) and Bernoulli's inequality $\left(1 + \frac{k-1}{\alpha}\right)^\alpha \geq 1 + \alpha \frac{k-1}{\alpha} = k$, and taking into account $\omega(n) = r$, $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r$ we arrive to (9).

Remark. For squarefree n one has $\omega(n) = \Omega(n) = r$, so (9) implies that $d_k(n) = k^r$ for such $n \cdot (k \geq 2)$.

III. A generalization. A generalization of $d_k(n)$ can be obtained by considering sums of the type

(11) $$A_k(n) = \sum_{n_1 n_2 \ldots n_k = n} a_{n_1} a_{n_2} \ldots a_{n_k}$$

where (a_n) is a sequence of real numbers such that the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $\mathrm{Re}\, s > 1$. For $a_n = 1 (n = 1, 2, 3, \ldots)$ one reobtains the function $d_k(n)$.

The following property can be proved similarly with (3):

(12) $$\textbf{1.} \quad \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right)^k = \sum_{n=1}^{\infty} \frac{A_k(n)}{n^s} = \mathcal{D}(A_k, s)$$

Let now $\varphi_k(n)$ denote the Jordan arithmetical function ([4], [6]) and $\sigma_k(n)$ be the sum of k-th powers of divisors of n. The following known relations: $\mathcal{D}(\varphi_k, s) = \zeta(s-k)/\zeta(s)$, $\mathcal{D}(\sigma_k, s) = \zeta(s-k) \cdot \zeta(s)$ and the identity $(\zeta(s-k)/\zeta(s))^k \cdot (\zeta(s))^{k+1} = (\zeta(s-k)\zeta(s))^k$ lead to

(13) $$(\mathcal{D}(\varphi_k, s))^k \cdot \mathcal{D}(d_{k+1}, s) = (\mathcal{D}(\sigma_k, s))^k$$

Let us introduce the function

(14) $$\Phi_l(n) = \sum_{n_1 n_2 \ldots n_k = n} \varphi_l(n_1)\, \varphi_l(n_2) \ldots \varphi_l(n_k)$$

and analogously

(15) $$\Sigma_l(n) = \sum_{n_1 n_2 \ldots n_k = n} \sigma_l(n_1)\, \sigma_l(n_2) \ldots \sigma_l(n_k).$$

Using the well-known Dirichlet product ([1], [3]) and the uniqueness theorem of Dirichlet series ([8]) from (12), (13) we get the relation:

(16) $$\textbf{2.} \quad \sum_{i \mid n} \Phi_k\left(\frac{n}{i}\right) d_{k+1}(i) = \Sigma_k(n) \text{ and } (k = 1)\ \varphi(n) \leqslant d(n) \cdot \sigma(n)$$

COROLLARY.

(17) $$\Sigma_k(n) \leqslant d_{k+1}(n) \cdot \sum_{i \mid n} \Phi_k(i)$$

Proof. This follows at once from (16, by using (4)).

IV. Asymptotic results. 1. *The number of solutions in positive integers of the inequation $x_1 x_2 \ldots x_k \leqslant n$ is given by the sum $\sum_{i=1}^{n} d_k(i)$.*

Proof. Divide all the systems $(x_1, x_2, \ldots, x_k)$, which satisfy the above inequation, in n groups ordered by $1, 2, 3, \ldots, n$. Attribute to the i-th group the system satisfying $x_1 x_2 \ldots x_k = i$. The number of these systems being $d_k(i)$, by summation we obtain the result.

In the light of 1 it would be interesting to study the sum $\sum_{i=1}^{n} d_k(i)$ or more generally $\sum_{n \leqslant x} d_k(n)$ with x a real number. We can prove:

(18) $$\textbf{2.} \quad \sum_{n \leqslant x} d_k(n) = \frac{1}{(k-1)!} x \log^{k-1} x + 0(x \log^{k-2} x),\ k \geqslant 2.$$

Proof. We shall use induction with respect to k. For $k = 2$, (18) becomes $\sum_{n \leqslant x} d(n) = x \log x + 0(x)$, which is well known ([1], [3]). Assume now that (18) is valid, and try to prove it for $k+1$. Applying (1) we have $\sum_{n \leqslant x} d_{k+1}(n) = \sum_{n \leqslant x} \sum_{i \mid n} d_{k-1}(i) = \sum_{t \leqslant x} \left[\frac{1}{(k-1)!} \frac{x}{t} \log^{k-1} \frac{x}{t} + 0\left(\frac{x}{t} \log^{k-2} \frac{x}{t}\right)\right]$. Let us consider first the sum $\sum_{t \leqslant x} \frac{1}{t} \log^{k-1} \frac{x}{t} = \sum_{t \leqslant x} \frac{1}{t} \Big[\log^{k-1} x - \binom{k-1}{1} \log^{k-2} x \log t + \binom{k-1}{2} \log^{k-3} x \log^2 t - \ldots + (-1)^{k-1} \log^{k-1} t \Big] = \log^{k-1} x \left(\log x + C + 0\left(\frac{1}{x}\right)\right) - \binom{k-1}{1} \log^{k-2} x \left(\frac{\log^2 x}{2} + 0\left(\frac{\log x}{x}\right)\right) + \binom{k-1}{2} \log^{k-3} x \cdot \left(\frac{\log^3 x}{3} + 0\left(\frac{\log^2 x}{x}\right)\right) - \ldots + (-1)^{k-1} \left(\frac{\log^k x}{k} + 0\left(\frac{\log^{k-1} x}{x}\right)\right)$. Here we have used $\sum_{t \leqslant x} \frac{1}{t} = \log x + C + 0\left(\frac{1}{x}\right)$, where C is Euler's constant, and $\sum_{t \leqslant x} \frac{\log t}{t} = \frac{\log^2 x}{2} + 0\left(\frac{\log x}{x}\right)$, etc. (see [1], [3]). The known identity $1 - \frac{1}{2}\binom{k-1}{1} + \frac{1}{3}\binom{k-1}{2} - \ldots + \frac{1}{k}(-1)^{k-1} = \frac{1}{k}$ the equality $\frac{1}{(k-1)!} \cdot \frac{1}{k} = \frac{1}{k!}$ and an analogous argument to the above for $\sum_{t \leqslant x} 0\left(\frac{x}{t} \log^{k-2} \frac{x}{t}\right)$ imply easily that the property (18) is indeed valid for $k+1$.

COROLLARY. ([5], p. 94, Aufgaben 8)

(19) $$\sum_{n \leqslant x} d_k(n) \sim \frac{1}{(k-1)!} x \log^{k-1} x \quad (x \to \infty)$$

For the maximum order of magnitude of $\log d_k(n)$ one has:

(20) $$\textbf{3.} \quad \overline{\lim_{n \to \infty}} \frac{\log d_k(n) \cdot \log \log n}{\log n} = \log k \quad (k \geqslant 2)$$

Proof. We need the following theorem due to Drozdova and Freiman (see [4] p. 125): Let $f(n)$ be a multiplicative arithmetical function with the property $f(p^\nu) = g(\nu)$, where p is prime number and $g(\nu)$ depends only on ν. Suppose $g(\nu) \geq 1$ and there exists ν_0 with $g(\nu_0) > 1$. Assume also that for a certain real number $a > 0$ one has $\log g(\nu) = 0(\nu^{1-a})$. Then the "maximum order of magnitude" of $\log f(n)$ is given by $\frac{\log g(m)}{m} \cdot \frac{\log n}{\log \log n}$ where m is definite by $\frac{\log g(\nu)}{\nu} \begin{cases} \leq \frac{\log g(m)}{m} & \text{for } \nu \leq m \text{ and} \\ < \frac{\log g(m)}{m} & \text{for } \nu > m. \end{cases}$

In our case we have $d_k(p^\nu) = \binom{k+\nu-1}{\nu} = g(\nu)$. Here $g(\nu) = \binom{k+\nu-1}{k-1} = \frac{(k+\nu-1)(k+\nu-2)\ldots\nu}{(k-1)!} < 2^{\sqrt{\nu}}$ (k fixed) for sufficiently large ν. Thus we can choose $a = 1/2$. The inequality (10) assures that $\binom{k+\nu-1}{\nu}^{1/\nu} \leq k = \binom{k+1-1}{1}$, so $\frac{\log g(\nu)}{\nu} \leq \frac{\log g(1)}{1}$ which means that $m = 1$. All conditions of the above theorem of Drozdova and Freiman are verified, thus the maximum order of magnitude of $\log d_k(n)$ is $\log k \cdot \log n/\log\log n$, i.e. our theorem.

4. *The "right" (or normal) order of magnitude of $\log d_k(n)$ is $\log k \cdot \log\log n$, i.e. for all $\varepsilon > 0$, the inequalities*

$$k^{(1-\varepsilon)\log\log n} < d_k(n) < k^{(1+\varepsilon)\log\log n} \tag{21}$$

are true for almost all n.

Proof. We use (9) and a result of Hardy and Ramanujan (see [3], [4]) which says that the right order of magnitude of $\omega(n)$ and $\Omega(n)$ is $\log\log n$. Thus we can write : $(1-\varepsilon)\log\log n < \omega(n) \leq \log d(n)/\log_k k \leq \Omega(n) < (1+\varepsilon)\log\log n$ for almost all n. This proves (21).

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