MATHEMATICA—REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 18, N° 1, 1989, pp. 95–100

GENERALIZATION OF THE NEWTONIAN MECHANICS' FUNDAMENTAL LAW

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Abstract. In the present paper a new mathematical formulation is given to the Newtonian mechanics' fundamental law, for the material point with variable mass. This new mathematical formulation includes the higher order accelerations, permitting their determination.

1. Introduction. In a previous published paper [5] I generalized the Newtonian mechanics' fundamental law

$$m\ddot{\bar{r}} = \bar{R}(t, \bar{r}, \dot{\bar{r}}),$$

for the material point with constant mass m, where t is the time, $\bar{r}$ — the instantaneous position vector of the material point (considered zero order acceleration), $\dot{\bar{r}}$ — the linear velocity (considered first order acceleration), $\ddot{\bar{r}}$ — the second order linear acceleration, and $\bar{R}$ — the resultant of forces applied to the considered material point.

In this paper a new mathematical formulation is given to the Newtonian mechanics' fundamental law, for the material point with variable mass, in which the higher order accelerations are included.

2. Mathematical formulation of the motion fundamental law. It is known that Newtonian mechanics' fundamental equation, for the variable mass point is [4]

$$(m\bar{v})^{\cdot} = \bar{R}(t, \bar{r}, \dot{\bar{r}}). \tag{1}$$

By determining the accelerations $\overset{(i)}{\bar{r}}$, it follows

$$\overset{(i)}{\bar{r}} = \sum_{\sigma=0}^{n-i-1} \overset{(i+\sigma)}{\bar{r}}(0)\frac{t^{\sigma}}{\sigma!} + \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}\,\bar{\varphi}_n(s)\,\mathrm{d}s, \tag{2}$$

where

$$\overset{(n)}{\bar{r}} = \bar{\varphi}_n(t), \quad (n = 3, 4, 5, \ldots), \ (i = 0, 1, 2.),$$

"n" being the higher order of the acceleration.

By substituting (2) for $i = 1, 2$ in (1), one obtains

$$\bar{R}(t, \bar{r}, \dot{\bar{r}}) - \int_0^t K_n(t, s)\,\bar{\varphi}_n(s)\,\mathrm{d}s = \bar{f}_n(t), \tag{3}$$

where

$$K_n(t, s) = m(t) \frac{(t-s)^{n-3}}{(n-3)!} + \dot{m}(t) \frac{(t-s)^{n-2}}{(n-2)!}, \tag{4}$$

$$\bar{f}_n(t) = \left[m(t) \sum_{\sigma=0}^{n-3} \overset{(2+\sigma)}{\bar{r}}(0) + \dot{m}(t) \sum_{\sigma=0}^{n-2} \overset{(1+\sigma)}{\bar{r}}(0) \right] \frac{t^\sigma}{\sigma!}. \tag{5}$$

Equation (3) represents a new mathematical formulation of the motion fundamental law. It is also called "integrodifferential fundamental equation of the Newtonian mechanics, for the material point with variable mass". Though different as form, equations (1) and (3) have equivalent semnifications. Both of them represent the basic law of the motion in the Newtonian mechanics. Equation (1), by the variable mass, sets in relation the force with the second order acceleration. Equation (3), by the variable mass, sets in relation the force with the higher order accelerations.

3. Particular cases

a) If $\bar{R} = \bar{R}(t)$, equation (3) becomes

$$\int_0^t K_n(t, s)\, \bar{\varphi}_n(s)\, ds = \bar{F}_n(t), \tag{6}$$

where

$$\bar{F}_n(t) = \bar{R}(t) - \bar{f}_n(t). \tag{7}$$

Expression (6) is "an integral equation of the first order linear Volterra type".

b) If $\bar{R}$ has the form

$$\bar{R}(t, \bar{r}, \dot{\bar{r}}) = \bar{A}(t) - a_0(t)\bar{r} - a_1(t)\dot{\bar{r}},$$

equation (3) has the expression

$$\int_0^t N_n(t, s)\, \bar{\varphi}_n(s)\, ds = \bar{E}_n(t), \tag{8}$$

in which

$$N_n(t, s) = K_n(t, s) + \sum_{\nu=0}^{1} a_\nu(t) \frac{(t-s)^{n-\nu-1}}{(n-\nu-1)!}, \tag{9}$$

$$\bar{E}_n(t) = \bar{A}(t) - \left[\bar{f}_n(t) + \sum_{\nu=0}^{1} \sum_{\sigma=0}^{n-\nu-1} a_\nu(t) \overset{(\nu+\sigma)}{\bar{r}}(0) \frac{t^\sigma}{\sigma!} \right]. \tag{10}$$

Like (6), expression (8) is "a Volterra linear integral equation of the first order".

4. Analytical solution of the Volterra integral equation (8). Equation (8) is reduced to a second kind linear equation, if the following conditions are fulfilled

$$N_n(t, t) \neq 0, \quad \bar{E}_n(0) = \bar{0}. \tag{11}$$

The first condition (11) not being fulfilled, equation (8) is derived $n-2$ times, in relation with the time, and one obtains the second kind integral equation

$$\bar{\varphi}_n(t) + \int_0^t N_n^1(t, s)\, \bar{\varphi}_n(s)\, ds = \bar{E}_n^1(t), \tag{12}$$

where

$$N_n^1(t, s) = [m(t)]^{-1} \frac{\partial^{n-2} N_n(t, s)}{\partial t^{n-2}}, \tag{13}$$

$$\bar{E}_n^1(t) = [m(t)]^{-1} \overset{(n-2)}{\bar{E}_n}(t). \tag{14}$$

Having

$$\left[\frac{\partial^{n-3} N_n(t, s)}{\partial t^{n-3}} \right]_{s=t} = m(t) \neq 0,$$

the first condition (11) is fulfilled.

Taking into account (10), the second condition (11) becomes

$$\overset{(n-3)}{\bar{A}}(0) - \overset{(n-3)}{\bar{f}_n}(0) - \left[\sum_{\nu=0}^{1} \sum_{\sigma=0}^{n-\nu-1} a_\nu(t) \overset{(\nu+\sigma)}{\bar{r}}(0) \frac{t^\sigma}{\sigma!} \right]_{t=0}^{(n-3)} = \bar{0}. \tag{15}$$

The initial conditions $\overset{(i)}{\bar{r}}(0)$, $(i = 0, 1)$ are arbitrary. The initial conditions for $i > 1$ are determined from the following equation

$$m(t)\, \ddot{\bar{r}} + [\dot{m}(t) + a_1(t)]\dot{\bar{r}} + a_0(t)\, \bar{r} = \bar{A}(t), \tag{16}$$

both directly and by derivation.

By applying the method of successive approximations, the solution of equation (12) is obtained under the form [3]

$$\bar{\varphi}_n(t) = \bar{\varphi}_{n,0}(t) + \bar{\varphi}_{n,1}(t) + \bar{\varphi}_{n,2}(t) + \ldots + \bar{\varphi}_{n,m}(t) + \ldots, \tag{17}$$

where

$$\bar{\varphi}_{n,0}(t) = \bar{E}_n^1(t),$$

$$\bar{\varphi}_{n,1}(t) = -\int_0^t N_n^1(t, s)\, \bar{\varphi}_{n,0}(s)\, ds,$$

. . . .

$$\bar{\varphi}_{n,m}(t) = -\int_0^t N_n^1(t, s)\, \bar{\varphi}_{n,m-1}(s)\, ds.$$

5. Numerical solution of the Volterra integral equation (8). The approximative solution of equation (8) is determined by means of a numerical integration. We apply on the interval $[0, a]$, $a > 0$, a method analogous to that of the polygonal lines. We divide the interval $[0, a]$ by the points $t_k = k\dfrac{a}{m}$, $k = \overline{1, m}$, and we consider the quadrature formula

$$\int_0^{k\frac{a}{m}} \bar f(s)\,\mathrm{d}s \approx \frac{a}{m}\sum_{\nu=1}^{k} \bar f\left(\nu\frac{a}{m}\right), \quad (k = 1, 2, \ldots, m). \tag{18}$$

By writing that equation (8) is verified for $t_k = k\dfrac{a}{m}$, and by using the formula (18) for the approximative calculation of the integral, one obtains an algebraic system of m equations with m unknown quantities

$$\frac{a}{m}\sum_{\nu=1}^{k} N_n\left(k\frac{a}{m}, \nu\frac{a}{m}\right)\bar\varphi_n\left(\nu\frac{a}{m}\right) = \bar E_n\left(k\frac{a}{m}\right), \quad (k = 1, 2, \ldots, m). \tag{19}$$

The unknown quantities of system (19) are

$$\bar\varphi_n\left(\frac{a}{m}\right),\ \bar\varphi_n\left(2\frac{a}{m}\right), \ldots, \bar\varphi_n(a).$$

In numerical values, the solution of system (19) is obtained by using the known methods [1].

6. Analytical solution of the Volterra integral equation (6) for $n = 3$. For $n = 3$, equation (6) becomes

$$\int_0^t K_3(t, s)\,\bar\varphi_3(s)\,\mathrm{d}s = \bar F_3(t), \tag{20}$$

where

$$\bar F_3(t) = \bar R(t) - \bar f_3(t). \tag{21}$$

Equation (20) is converted into second kind integral equation

$$\bar\varphi_3(t) + \int_0^t K_3^1(t, s)\,\bar\varphi_3(s)\,\mathrm{d}s = \bar F_3^1(t), \tag{22}$$

in which

$$K_3^1(t, s) = [m(t)]^{-1}\frac{\partial K_3(t, s)}{\partial t}, \tag{23}$$

$$\bar F_3^1(t) = [m(t)]^{-1}\dot{\bar F}_3(t). \tag{24}$$

For $n = 3$, we have

$$K_3(t, s) = m(t) + \dot m(t)\,(t - s), \tag{25}$$

$$\bar F_3(t) = \bar R(t) - \dot{\bar r}(0)\,\dot m(t) - \ddot{\bar r}(0)[m(t) + t\,\dot m(t)], \tag{26}$$

and relations (23) and (24) become

$$K_3^1(t, s) = [m(t)]^{-1}[2\dot m(t) + \ddot m(t)\,(t - s)], \tag{27}$$

$$\bar F_3^1(t) = [m(t)]^{-1}\{\dot{\bar R}(t) - 2\ddot{\bar r}(0)\,\dot m(t) - [\dot{\bar r}(0) + \ddot{\bar r}(0)t]\,\ddot m(t)\}. \tag{28}$$

From (25) and (26) conditions (11) result

$$K_3(t, t) = m(t) \neq 0, \tag{29}$$

$$\bar R(0) - \dot m(0)\,\dot{\bar r}(0) - m(0)\,\ddot{\bar r}(0) = \bar 0. \tag{30}$$

The solution of equation (22) is of the form

$$\bar\varphi_3(t) = \bar\varphi_{3,0}(t) + \bar\varphi_{3,1}(t) + \bar\varphi_{3,2}(t) + \ldots + \bar\varphi_{3,m}(t) + \ldots, \tag{31}$$

where

$$\bar\varphi_{3,0}(t) = \bar F_3^1(t),$$

$$\bar\varphi_{3,1}(t) = -\int_0^t K_3^1(t, s)\,\bar\varphi_{3,0}(s)\,\mathrm{d}s,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\bar\varphi_{3,m}(t) = -\int_0^t K_3^1(t, s)\,\bar\varphi_{3,m-1}(s)\,\mathrm{d}s.$$

7. Application. The material point with variable mass is considered in upward vertical motion, in the gravitational uniform field (g = const.) with the initial velocity $\bar v_0$. By admitting the capture of meteorites falling on the point, uniformly, the Newtonian equation, proposed by T. Levi-Civita, for the case of the capture, is [4]

$$(m\bar v)^{\cdot} = \bar R. \tag{32}$$

We shall take, as a mass variation law, the expression

$$m(t) = m_0(1 - \alpha t), \tag{33}$$

where m_0 and α are constant.

For $n = 3$, equation (32) becomes (22).

By deriving the law (33) in relation with the time, we have successively

(34) $$\dot{m}(t) = -\alpha m_0, \quad \ddot{m}(t) = 0.$$

By considering

$$\bar{R}(t) = m(t)\,\bar{g}, \quad (g = \text{const.}),$$

we have

(35) $$\dot{\bar{R}}(t) = -\alpha m_0 \bar{g}.$$

By observing (34), condition (30) becomes

$$m_0[\bar{g} + \alpha\bar{v}_0 - \ddot{\bar{r}}(0)] = \bar{0},$$

whence it follows

$$\ddot{\bar{r}}(0) = \bar{g} + \alpha\bar{v}_0, \quad (m_0 \neq 0).$$

By taking into account (34) and (35), expressions (27) and (28) become

$$K_3^1(t, s) = \frac{2\alpha}{\alpha t - 1}, \quad \bar{F}_3^1(t) = \frac{\alpha(2\alpha\bar{v}_0 + \bar{g})}{1 - \alpha t}.$$

By observing (31), the solution of equation (22) in the second approximation is

$$\bar{\varphi}_3(t) \approx \bar{\varphi}_{3,0}(t) + \bar{\varphi}_{3,1}(t).$$

By effectuating the calculi, the third order acceleration of the point with variable mass has the expression

(36) $$\bar{\varphi}_3(t) \approx \alpha(1 - \alpha t)^{-1}[1 - 2\ln(1 - \alpha t)](2\alpha\bar{v}_0 + \bar{g}).$$

By projecting (36) on the upward verical axis (Oz), it follows

$$\dddot{z}(t) \approx \alpha(1 - \alpha t)^{-1}[1 - 2\ln(1 - \alpha t)](2\alpha v_0 - g).$$

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Received 17 IV 1986

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