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SOME EXTENSIONS OF THE ABSTRACT DINI CONVERGENCE THEOREM

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I. Introduction. The classical Dini convergence theorem reads as follows:

1. THEOREM. *If $T \subset \mathbb{R}$ is a compact interval and $(x_n)_{n \in \mathbb{N}}$ is a monotonic sequence of continuous functions $x_n : T \to \mathbb{R}$ converging pointwise to a continuous function $x : T \to \mathbb{R}$ then (x_n) converges uniformly on T to x.*

There are many extensions of this theorem where the monotonicity is generated by a cone (as in [5]) or where the interval T is replaced by a topological space (compact, countably compact, pseudocompact). A detailed presentation of these aspects is to be found in a recent paper of I. Muntean [3].

An abstract version of the Dini theorem in the context of ordered spaces appears in the H. H. Schaefer's treatise [7]. Recall that if E is a real ordered locally convex vector space, the positive cone E_+ is normal if and only if there exists a family of seminorms $\mathscr{P}$ generating the topology of E such that $p(x) \leqslant p(x + y)$ for each $p \in \mathscr{P}$ and $x, y \in E_+$ (Such seminorms are called monotone).

2. THEOREM. [7]. *Let E be a real ordered separated locally convex vector space, whose positive cone E_+ is normal and suppose $S \subseteq E$ directed for $\leqslant$. If the section filter of S (i.e. the filter generated by the base $\{\{y \in S : y \geqslant x\} : x \in S\}$) converges in the weak topology $\sigma(E, E')$, then it converges in E.*

Note that the abstract version of Dini theorem implies the classical one: in fact it T is a compact topological space then $E = C(T)$ is a real normed ordered space with a normal positive cone (E is even a Banach lattice); the section filter of an increasing sequence (x_n) has the basis $\{x_k : k \geqslant n\}_{n \in \mathbb{N}}$ and the weak convergence in $C(T)$ is equivalent to the pointwise convergence and uniform boundedness.

The purpose of this note is to obtain refinements of the abstract Dini theorem for sequences.

II. Convergence results of Rădulescu and Alexandrov type. Let E be a real ordered separated locally convex vector space. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in E satisfies the Rădulescu condition ([6], [3]) if there exists x in E such that $x_n \xrightarrow{\sigma(E,E')} x$ $(n\to\infty)$ and for any positive integers m and n, x_m and x_n are comparable (i.e. $x_n \leqslant x_m$ or $x_m \leqslant x_n$). Here E' denotes the topological dual of E. We need the following lemma (see [1] p. 92 for E a normed space; the proof is still valid for the locally convex case)

1. LEMMA. *Let E be a real ordered separated locally convex vector space. The following statements are equivalent:*

(i) *E_+ is closed.*

(ii) *For every x_0 in E, if $f(x_0)\geqslant 0$ for each f in E'_+ then $x_0 \geqslant 0$.*

(E'_+ denotes the dual cone $\{f\in E' : f(E_+)\subseteq \mathbb{R}_+\}$)

2. THEOREM. *Let E be a real ordered separated locally convex vector space whose positive cone E_+ is normal. If the sequence $(x_n)_{n\in\mathbb{N}}$ in E satisfies the Rădulescu condition then $x_n\to x$ $(n\to\infty)$.*

Proof. We may suppose that E_+ is closed. In fact $\bar{E}_+$ is a normal cone ([7] p. 216). Note that it is also proper i.e. $\bar{E}_+ \cap -\bar{E}_+ = \{0\}$ since E is separated.

Denote $y_n = x_n - x$ $(n\in\mathbb{N})$. We shall prove that for any n in $\mathbb{N}$, $y_n\geqslant 0$ or $y_n\leqslant 0$. Suppose by contradiction that there exists n_0 in $\mathbb{N}$ such that $y_{n_0}\ngeqslant 0$ and $y_{n_0}\nleqslant 0$. Applying the preceding lemma there exist f, g in E'_+ with $f(y_{n_0})<0$ and $g(y_{n_0})>0$.

Because $\lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} g(y_n) = 0$ there exists n in $\mathbb{N}$ with $f(y_n) > f(y_{n_0})$ and $g(y_n) < g(y_{n_0})$. Applying the lemma again one obtains $y_n \nleqslant y_{n_0}$ and $y_n \ngeqslant y_{n_0}$ contradicting the fact that y_n and y_{n_0} are comparable.

We may suppose $y_n\geqslant 0$, $y_n\neq 0$ for every n in $\mathbb{N}$. (In fact, writing $N_+ = \{n\in\mathbb{N} : y_n\geqslant 0,\ y_n\neq 0\}$, $N_- = \{n\in\mathbb{N} : y_n\leqslant 0,\ y_n\neq 0\}$, $N_0 = \{n\in\mathbb{N} : y_n = 0\}$ we have shown that $\mathbb{N} = N_+\cup N_-\cup N_0$. If the conclusion of the theorem holds for the subsequence $(x_n)_{n\in N_+}$ it will also hold for $(x_n)_{n\in N_-}$, replacing y_n with $-y_n$.)

For m in $\mathbb{N}$, write $N_m = \{n\in\mathbb{N} : y_m\leqslant y_n\}$. The set N_m is finite: in fact if N_m is infinite then there exist $n_1<n_2<\dots$, $n_i\in N_m$ $(i\in\mathbb{N})$, but for each $f\in E'_+$ we have $0\leqslant f(y_m)\leqslant f(y_{n_i})\to 0$ $(i\to\infty)$, hence $f(y_m)=0$, a contradiction with the lemma since $y_m\neq 0$.

We construct inductively $n_1 = 1$, $n_{k+1}\in \mathbb{N}\setminus\bigcup_{i=1}^{k} N_i$ with $n_{k+1}>n_k$. Then $y_{n_1}\geqslant y_{n_2}\geqslant\dots$ and $\sigma(E,E')-\lim\limits_{k\to\infty} y_{n_k} = 0$. Applying the abstract Dini theorem (I.2) for the sequence $(y_{n_k})_{k\in\mathbb{N}}$ one obtains $y_{n_k}\to 0$ $(k\to\infty)$. We complete the proof showing that $\lim\limits_{n\to\infty} y_n = 0$. Let $p\in\mathcal{P}$ be a monotone seminorm and $\varepsilon>0$. Since $\lim\limits_{k\to\infty} p(y_{n_k}) = 0$, there exists $k_\varepsilon\in\mathbb{N}$ such that $p(y_{n_k})\leqslant\varepsilon$ for any $k\geqslant k_\varepsilon$. N_{k_ε} is finite so there exists n_ε in $\mathbb{N}$ such that $y_n\leqslant y_{k_\varepsilon}$ for $n\geqslant n_\varepsilon$. For $n\geqslant n_\varepsilon$ we have $0\leqslant y_n\leqslant y_{k_\varepsilon}$, hence $p(y_n)\leqslant p(y_{k_\varepsilon})\leqslant\varepsilon$.

In conclusion $y_n\to 0$, $x_n\to x$ $(n\to\infty)$. q.e.d.

3. REMARKS. a) The condition on the normality of the positive cone cannot be dropped: there exist Banach spaces with a non-normal positive cone for which the Dini property (for monotonic sequences) does not hold. For example $E = l^2$, $E_+ = \{0\}\cup\{x = (x_1, x_2,\dots)\in l^2 : \exists N\in\mathbb{N},\ x_N > 0 \text{ and } x_n = 0,\ \forall n > N\}$. E misses the Dini property since the sequence $x^n = (0,\dots,0,\underset{n}{1},0,\dots)$ $(n\in\mathbb{N})$ is increasing, weakly convergent to 0 but $\|x^n\| = 1$.

b) On the other hand E may have the Dini property (or even the more general property using the Rădulescu condition) without the positive cone being normal. In fact, for the Banach space $E = l^1$ the norm and weak sequential convergence are the same and it is sufficient to exhibit a non-normal cone in E, but such a construction is possible in every infinite dimensional Banach space (cf. [4]).

c) A. B. Nemeth [4], proved that an ordered Banach space in which every weakly convergent series with positive terms is unconditionally convergent has its positive cone normal.

The monotonicity condition in the Dini theorem was relaxed in [2], [3] using an Alexandrov condition like:

$$\sum_{i=0}^{k} c_i|x(t) - x_{n+1+k-i}(t)|^{r_i} \leqslant \sum_{i=0}^{k} c_i|x(t) - x_{n+k-i}(t)|^{r_i}, \quad (t\in T,\ n\in\mathbb{N})$$

where $k\in\mathbb{N}$, $c_i, r_i > 0$ $(i = 0,1,\dots,k)$ are fixed.

In the sequel we shall obtain an abstract Dini theorem replacing the monotonicity of the sequence (x_n) with an Alexandrov type condition.

Let E, F be real ordered locally convex vector spaces. We shall say that a sequence of mappings $\varphi_n : E^{\mathbb{N}}\to F$ is an Alexandrov sequence if the following conditions are true:

1° For each positive integer n, $\varphi_n(\bar{0}) = 0$ and φ_n is "increasing" in the sense: $\bar{x} = (x_1, x_2,\dots)\in E_+^{\mathbb{N}}$, $\bar{y} = (y_1,y_2,\dots)\in E_+^{\mathbb{N}}$ with $\bar{x}\leqslant\bar{y}$ (coordinatewise) implies $\varphi_n(\bar{x})\leqslant\varphi_n(\bar{y})$.

2° If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E weakly convergent to 0 then $\sigma(F,F')\text{-}\lim\limits_{n\to\infty}\varphi_n(\bar{x}) = 0$, $(\bar{x} = (x_1,x_2,\dots)\in E^{\mathbb{N}})$.

3° There exist V and W neighbourhoods of the origin in E and F and there exists a positive integer k such that for each n in $\mathbb{N}$ the mapping $\psi_n : x\in V\cap E_+ \to \varphi_n(0,0,\dots,0,\underset{n+k}{x},0,\dots)\in W\cap F_+$ is a bijection and the family of functions $\{\psi_n^{-1} : n\in\mathbb{N}\}$ is equicontinuous in 0.

4. THEOREM. *Let E, F be real ordered separated locally convex vector spaces, F having a normal positive cone and let (φ_n) be an Alexandrov sequence. If (x_n) is a sequence in E_+ weakly convergent to 0 and for any m, n in $\mathbb{N}$ one has $\varphi_n(\bar{x})\leqslant\varphi_m(\bar{x})$ or $\varphi_m(\bar{x})\leqslant\varphi_n(\bar{x})$ where $\bar{x} = (x_1,x_2,\dots)$, then $x_n\to 0$ $(n\to\infty)$ in E.*

Proof. Denote $y_n = \varphi_n(\bar{x})$. The sequence (y_n) satisfies the Rădulescu condition in F (cf. 1°, 2°) and the theorem 2 yields $y_n\to 0$. We have $0\leqslant\varphi_n(0,0,\dots,0,x_{n+k},0,\dots)\leqslant y_n$ and using the normality of the cone F_+ we get $z_n := \varphi_n(0,\dots,0,x_{n+k},0,\dots) = \psi_n(x_{n+k})\to 0$ $(n\to\infty)$.

Using now the condition 3° we have $x_{n+k} = \psi_n^{-1}(z_n) \to 0$ $(n \to \infty)$ and so $x_n \to 0$ $(n \to \infty)$. q.e.d.

5. APPLICATIONS. a) Let E be a real ordered separated locally convex vector space whose positive cone is normal and (x_n) a sequence in E_+ weakly convergent to 0 such that

$$x_{n+2} \leqslant 1/2(x_{n+1} + x_n), \ (n \in \mathbb{N}) \qquad (*)$$

Then $x_n \to 0 (n \to \infty)$.

In fact, choosing in the theorem $\varphi_n(x_1, x_2, \ldots) = x_{n+1} + 1/2\ x_n$, (*) can be written as $\varphi_{n+1}(\bar{x}) \leqslant \varphi_n(\bar{x})$ hence (φ_n) is an Alexandrov sequence.

In the paper [2] this result is obtained by another method for E a normed lattice. Our proof shows that E need not be a lattice.

Note that the condition (*) cannot be relaxed to:

$$x_{n+2} \leqslant 1/2(x_{n+1} + x_n) \ \text{ or } \ x_{n+2} \geqslant 1/2(x_{n+1} + x_n), \ (n \in \mathbb{N}) \qquad (*')$$

For a counterexample it is sufficient to take $E = C[0,1]$, (y_n) a bounded sequence in E which converges pointwise but not uniformly (e.g. $y_n(t) = t_n - t^{2n}$, $t \in [0, 1]$) and

$$x_{3n} = y_n, \ x_{3n-1} = x_{3n-2} = 0 \ (n \in \mathbb{N}).$$

The same example shows that in theorem 2, the Rădulescu condition cannot be relaxed by demanding that each consecutive elements be comparable.

b) However (*) can be replaced by

$$x_{n+2} \leqslant (1 - t_{n+1})x_{n+1} + t_n x_n \ (n \in \mathbb{N}) \qquad (**)$$

where (t_n) is a bounded sequence of positive real numbers, by choosing in the preceding theorem $\varphi_n(\bar{x}) = x_{n+1} + t_n x_n$.

To obtain a convergence result to an element x $(\neq 0)$ we apply the preceding theorem for the sequence $(|x - x_n|)_{n \in \mathbb{N}}$:

6. THEOREM. *Let E be a locally convex vector lattice, F an ordered separated locally convex vector space having a normal positive cone and (φ_n) an Alexandrov sequence. Let (x_n) be a sequence in E weakly convergent to x so that for any i, j in $\mathbb{N}$ one has*

$$\varphi_i((|x - x_n|)_{n \in \mathbb{N}}) \leqq \varphi_j((|x - x_n|)_{n \in \mathbb{N}}).$$

Then $x_n \to x$ $(n \to \infty)$.

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