MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 18, N° 1, 1989, pp. 19—26

ON QUADRATIC EQUATIONS

IOANNIS K. ARGYROS
(Las Cruces)

Abstract. A new iteration is presented for finding solutions of the quadratic equation in a Banach space. Our results can apply to quadratic integral equations arising in the theories of radiative transfer, neutron transport and in the kinetic theory of gases.

Introduction. In the theories of radiative transfer and neutron transport [3], [4], [5], [11] an important role is played by nonlinear integral equations of the form

(1) $$x(s) = g(s) + x(s)\int_0^1 f(s,t)\,x(t)\,dt,$$

where $g(s)$ and $f(s, t)$ are given functions on $[0, 1]$.

Equation (2) can be considered as a special case of the equation

(2) $$x = y + x\tilde{K}(x)$$

where $\tilde{K}$ is a linear operator on a Banach algebra X_A and $y \in X_A$ is fixed. Obviously, equation (2) reduces to (1) if we take $y = g(s)$ and

$$\tilde{K}(x)(s) = \int_0^1 f(s,t)x(t)\,dt.$$

The method of successive substitutions, [1], [10], the continued fraction iteration, [9] and the Newton-Kantorovich iteration [6], [8] have been used to obtain a solution x^* of special cases of (1) (or (2)).

In almost all the above cases however the solution x^* satisfies the estimate

(3) $$\|x^*\| \leqslant \frac{1-\sqrt{1-4\|y\|b}}{2b} \equiv d,$$

where

$$b = \sup_{0 \leqslant s \leqslant 1}\int_0^1 |f(s,t)|\,dt$$

provided that

(4) $$4\|y\|b < 1.$$

Under the above assumption however it is known that the corresponding real quadratic equation has two solutions. We wonder if this can be true in a Banach space X. It turns out that this is true under certain assumptions. What we really need to do (assuming that (4) holds) is to generate an iteration $\{x_n\}$ convergent to a solution x^* of (1) (or (2)) which guarantees that if $\|x_0\| \geq d$ then $\|x_n\| \geq d$ and therefore

(5) $$\|x^*\| \geq d.$$

We suggest the iteration

(6) $$x_{n+1} = (L(x_n))(K(x_n))$$

for solving (1) (or (2)), where

(7) $$L(x) = x - y$$

and

(8) $$K(x) = \frac{1}{\tilde{K}(x)}$$

provided that $K(x)$ is well defined and $L(x) \neq 0$ on $U(z, r) = \{x \in X_A / \|x - z\| < r\}$ for some $z \in X_A$ and $r > 0$.

In the first part of this paper we give conditions for the convergence of (6) to a solution of (1) (or (2)) without making use of the standard hypothesis (4).

In the second part we provide conditions for the solution of the abstract quadratic equation

(9) $$x = y + B(x, x)$$

where B is a bounded bilinear operator on a Banach space X and $y \in X$ is fixed, using the iteration

(10) $$x_{n+1} = B(x_n)^{-1}(x_n - y),\ n = 0, 1, 2, \ldots$$

for some $x_0 \in X$.

Moreover we show that (10) has the property (5) if

(11) $$4\|B\| \cdot \|y\| < 1.$$

Finally note that for $B(w, v) = w\tilde{K}v$ and $X = X_A$ equation (9) reduces to (2).

I. Basic Results. We denote by $C[0, 1]$ the Banach space of all real continuous functions on $[0, 1]$ with the maximum norm,

(12) $$\|x\|_C = \max_{0 \leq s \leq 1} |x(s)|.$$

Note that the space $X_A = C[0,1]$ with norm given by (12) is a Banach algebra. In the rest of this part $\|x\|$ denotes $\|x\|_C$.

We can now prove a consequence of the contraction mapping principle theorem [11].

THEOREM 1. *Assume*:

(*i*) *there exist* $z \in X$ *and an interval* $I = [r_1, r_2)$, $r_1 > 0$ *such that the operator* T *given by*

(13) $$T(x) = (L(x))(K(x)),$$

where L *and* K *are given by* (7) *and* (8) *respectively is well defined, the operator* $K(z)$ *is bounded and a number* $a > 0$ *is given by*

(14) $$a = a(z, r) = \frac{\|K(z)\|}{1 - \|\tilde{K}\|\,\|K(z)\| \cdot r}$$

for

$$0 < r < \frac{1}{\|\tilde{K}\| \cdot \|K(z)\|},$$

(*ii*) *for any* $x \in I$ *the following inequalities are satisfied*:

(15) $$\|\tilde{K}\|(r + \|z - y\|)a^2 + a - 1 < 0;$$

(16) $$a\|I - z\tilde{K}\| - 1 < 0;$$

and

(17) $$\frac{a\|P(z)\|}{1 - a\|I - z\tilde{K}\|} \leq r < \|z - y\|$$

where

(18) $$P(z) = z\tilde{K}(z) + y - z.$$

Then

(*a*) *Equation* (2) *has a unique solution* $x^* \in U(z, r_2)$ *which can be obtained as the limit of the iteration*

$$x_{n+1} = (L(x_n))(K(x_n))$$

for any $x_0 \in U(z, r_2)$.

(*b*) *Moreover* $x^* \in \bar{U}(z, r_1)$.

Proof. Let $r \in I$ and choose $w, v \in \bar{U}(z, r)$.

Claim 1. T is a contraction on $\bar{U}(z, r)$.

We have,

(19) $$T(w) - T(v) = K(w)(w - y) - K(v)(v - y)$$
$$= K(v)\tilde{K}(v - w)K(w)(w - y) + K(v)(w - v).$$

Hence,

$$\|T(w) - T(v)\| \leq [\|\tilde{K}\|(r + \|z - y\|)a^2 + a]\|w - v\|.$$

The above inequality now and (15) justify the claim.

Claim 2. T maps $\bar{U}(z, r)$ *into* $\bar{U}(z, r)$.

The claim easily follows from the inequality

$$\|T(w) - z\| \leqslant a(\|I - z\bar{K}\|r + \|P(z)\|) \leqslant r,$$

using (16) and (17).

Remarks (a) The condition $r \leqslant \|z - y\|$ is imposed because otherwise if say, $x_0 = y \in \bar{U}(z, r)$ the sequence given by (6) is not defined.

(b) If the sequence $\{x_n\}$ generated by (6) is either increasing or decreasing and the rest of the hypotheses in Theorem 1 hold except (15) then the sequence $\{x_n\}$ is contained in $\bar{U}(z, r)$ and by the monotone convergence theorem, there exists $x^* > 0$, $x^* \in X_A$ such that $x_n \to x^*$ as $n \to \infty$. Since $x_{n+1} = (L(x_n))(K(x_n))$ and $x_n \to x^*$ it follows that x^* is a solution of equation (2) which may not be unique in $\bar{U}(z, r)$, since T may not be contraction operator on $\bar{U}(z, r)$.

We will now extend our results to include equation (9) and iteration (10).

II. Extension — Remarks. From now on we assume that X is a Banach space and that B in (9) is a bounded symmetric bilinear operator [1], [11]. The operator B is assumed to be symmetric without loss of generality since B can always be replaced by the mean $\bar{B}$ of B defined by

$$\bar{B}(x, y) = \frac{1}{2}(B(x, y) + B(y, x)), \quad x, y \in X.$$

We have

$$\bar{B}(x, x) = B(x, x) \text{ for all } x \in X.$$

Denote by $B(x)$, $x \in X$ the linear operator on X defined by

$$B(x)(y) = B(x, y), \; x, y \in X.$$

We are now going to show that iteration $\{x_n\}$ given by (10) in case of convergence to a solution x^* of (10) is such that $\|x^*\| \geqslant d$ under certain assumptions.

PROPOSITION 2. *Assume:*

(1) *The iteration*

$$x_{n+1} = B(x_n)^{-1}(x_n - y)$$

is well defined for all $n = 0, 1, 2, \ldots$ *for some* $x_0 \in X$ *and converges to a solution* x *of* (10).

(2) *The following is true:*

$$1 - 4\|B\| \cdot \|y\| > 0,$$

and

(3) *let* $p \in [p_1, p_2]$, *where* p_1, p_2 *are the solutions of the equation*

$$\|B\|p^2 - p + \|y\| = 0.$$

If,

$$\|x_0\| > p$$

then

$$\|x_n\| \geqslant p, \quad n = 0, 1, 2, \ldots$$

and

$$\|x\| \geqslant p.$$

(Note that d given by (3) for $b = \|B\|$ is such that $d \in [p_1, p_2]$).

Proof. Using (9) we have

$$B(x_n, x_{n+1}) = x_n - y$$

or,

$$\|x_n - y\| = \|B(x_n, x_{n+1})\| \leqslant \|B\| \cdot \|x_n\| \cdot \|x_{n+1}\|,$$

so,

$$\|x_{n+1}\| \geqslant \frac{\|x_n - y\|}{\|B\| \cdot \|x_n\|} \geqslant \frac{\|x_n\| - \|y\|}{\|B\| \cdot \|x_n\|}.$$

Assume that $\|x_k\| \geqslant p$ for all $k = 0, 1, 2, \ldots, n$. Since $\|x_n\| \geqslant p \geqslant \|y\|$ to show $\|x_{n+1}\| \geqslant p$, it is enough to show

$$\frac{\|x_n\| - \|y\|}{\|B\| \|x_n\|} \geqslant p$$

or

$$\|x_n\| \geqslant \frac{\|y\|}{1 - p\|B\|}.$$

Finally it suffices to show

$$p \geqslant \frac{\|y\|}{1 - p\|B\|}$$

or

$\|B\|p^2 - p + \|y\| \leqslant 0$ which is true for $p \in [p_1, p_2]$.

That completes the proof of the proposition.

Using the Banach lemma for the invertibility of linear operators [11] we can easily show the following result.

PROPOSITION. *Let* $z \in X$ *be such that the linear operator* $B(z)$ *is invertible. Then* $B(x)$ *is also invertible for all* $x \in U(z, R_0)$, *where*

$$R_0 = \frac{1}{\|B\| \cdot \|B(z)^{-1}\|}.$$

We will need the definition:

DEFINITION. Let $z \in X$ be such that the linear operator $B(z)$ is invertible. Let $R > 0$ be fixed and $R < R_0$.

The operators $\tilde{P}$, $\tilde{T}$ given by

$$\tilde{P}(x) = B(x, x) + y - x$$

and

$$\tilde{T}(x) = (B(x))^{-1}(x - y)$$

are then well defined on $U(z, R)$.

Define the real functions F_1 and F_2 on $\mathbb{R}^+$ by

$$F_1(R) = e_1R^2 + e_2R + e_3$$

and

$$F_2(R) = e_4R^2 + e_5R + e_6,$$

where

$$e_1 (\|B\| \cdot \|B(z)^{-1}\|)^2,$$

$$e_2 = -2\|B\| \cdot \|B(z)^{-1}\|,$$

$$e_3 = 1 - \|B(z)^{-1}\| - \|B\| \cdot \|B(z)^{-1}\|^2\|z - y\|,$$

$$e_4 = \|B\| \cdot \|B(z)^{-1}\|,$$

$$e_5 = \|B(z)^{-1}(I - B(z))\| - 1$$

and

$$e_6 = \|B(z)^{-1}\tilde{P}(z)\|.$$

Working as in Theorem 1 we can easily show the following consequence of the contraction mapping principle [11].

THEOREM 3. *Assume:*
(1) *there exists $z \in X$ such that the linear operator $B(z)$ is invertible.*
(2) *The following are true:*

$$e_3 > 0,$$

$$e_5 < 0,$$

$$e_5^2 - 4e_4e_6 > 0$$

and

(3) *there exists $R > 0$ such that*

$$F_1(R) > 0,$$

$$F_2(R) \leq 0$$

and

$$R < \|z - y\|.$$

Then

(a) *the operator $\tilde{T}$ given by*

$$\tilde{T} = B(x)^{-1}(x - y)$$

is well defined and it has a unique fixed point $x \in \bar{U}(z, R)$.

(b) *The iteration*

$$x_{n+1} = B(x_n)^{-1}(x_n - y), \quad n = 0, 1, 2, \ldots$$

is well defined and it converges to x for any $x_0 \in \bar{U}(z, R)$.

Moreover, if

$$1 - 4\|B\| \cdot \|y\| > 0$$

and

$$\|x_0\| > \frac{1}{2\|B\|}$$

then

$$\|x\| > \frac{1}{2\|B\|}.$$

Remarks 2. (a) If the hypotheses of Theorem 2 are true then equation (9) has two solutions x_1 and x_2 such that

$$\|x_1\| \leq d$$

and

$$\|x_2\| > d.$$

(b) If $X = X_A$, then the hypotheses of Theorem 1 can easily be verified. If X is a Banach space then the conditions of Theorem 2 may be difficult to verify since the invertibility of the linear operator $B(z)$ may be almost impossible. Moreover z has to be chosen close to the solution.

However the other two popular methods for solving (9), namely Newton's method

$$x_{n+1} = x_n - (2B(x_n) - I)^{-1}(\tilde{P}(x_n)), \quad n = 0, 1, 2, \ldots \tag{20}$$

and the method of succesive substitutions

$$x_{n+1} = y + B(x_n, x_n), \quad n = 0, 1, 2, \ldots \tag{21}$$

share similar difficulties.

In particular Newton's method also requires z to be "close" to the solution and the invertibility of the operator $I - 2B(x_n)$ at each step (or the invertibility of $(I - 2B(x_0))$ if we are referring to the modified Newton's method).

Moreover the method of successive substitution makes no use of the invertibility of the linear operator $B(z)$, but z must still be close to the solution and

$$\|z\| < d,$$

under hypothesis (11) [1], [2], [10]. Therefore it cannot be used to find a solution x such that

$$\|x\| > d,$$

since the solution obtained then satisfies

$$\|x\| \leq d.$$

Finally note that in a general Banach space X neither (20) nor (21) share the property of keeping the iterates away from zero as does iteration (10). Therefore iteration (10) if applicable can be used to find the "large" solutions of (9) (if they exist) under hypothesis (11).

REFERENCES

1. Argyros, I. K., *Quadratic equations and applications to Chandrasekhar's and related equation*, Bull. Austral. Math. Soc., **32** (1985), 2, 275–292.
2. Argyros, I. K., *On a class of nonlinear integral equations arising in neutron transport*, Aequationes Mathematicae, **35** (1988), 29–49.
3. Cahlon, B., *Numerical solutions of nonlinear Volterra integral equations*, J. Comput. Appl. Math., **7** (1981), 2, 121–128.
4. Case, K. M., Zweifel, P. F., *Linear transport theory*, Addison Wesley Publ. Reading, MA, 1967.
5. Chandrasekhar, S., *Radiative transfer*, Dover, Publ., New York, 1960.
6. Kelley, C. T., *Solution of the Chandrasekhar H-equation by Newton's method.*, J. Math. Phys., **21** (1980), 1625–1628.
7. Kuratowski, C., *Sur les espaces complets*, Fund. Math., **15** (1930), 301–309.
8. Legget, R. W., *On certain nonlinear integral equations*, J. Math. Anal. Appl., **57** (1977), 462–468.
9. McFarland, J. E., *An iterative solution of the quadratic equation in Banach spaces*, Proc. Amer. Math. Soc., **9** (1958), 824–830.
10. Rall, L. B., *Quadratic equations in Banach space*, Rend. Circ. Mat. Palermo, **10** (1961), 314–332.
11. Rall, L. B., *Computational solution of nonlinear operator equations*, John Wiley Publ., New York, 1968.

Received 20 VII 1988

Department of Mathematics
New Mexico State University
Las Cruces, NM 88003