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LOCAL REPRESENTATION OF DISTANCE FUNCTIONAL ON SMOOTH NORMED LINEAR SPACES

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Abstract. The main purposes of this paper are to give some local representation theorems for the distance functional e_G defined on smooth normed linear spaces in terms of semi-inner product in the sense of Lumer [5] and Tapia [9] and to give some characterizations of reflexivity for smooth Banach spaces.

0. Introduction. 1. Let $(X, \|\cdot\|)$ be a real or complex normed linear space and G a linear subspace in X. We can define the following real functional :

(0.1) $$e_G(x) := \inf_{g\in G} \|x - g\| = d(x, G), \quad x \in X;$$

which will be called the distance functional associated to linear subspace G.

The following properties of the distance functional e_G were established by Miron Nicolescu [6] in 1938 (see also [8] pp. 137).

0.1. THEOREM. (M. Nicolescu). *Let $(X, \|\cdot\|)$ be a normed linear space and G its linear subspace. Then the following sentences are valid :*

(i) *For all* $x \in X$:

(0.2) $$0 \leq e_G(x) = e_{\bar{G}}(x) < \infty;$$

(ii) *If G_1 is a linear subspace in G, then*

(0.3) $$e_G(x) \leq e_{G_1}(x), \quad x \in X;$$

(iii) *We have*

(0.4) $$e_G(d) = 0 \ \text{if} \ d \in G;$$

(0.5) $$e_G(x + g) = e_G(x) + e_G(g) = e_G(x), \quad x \in X \text{ and } g \in G;$$

(0.6) $$e_G(x + y) \leq e_G(x) + e_G(y), \quad x, y \in X;$$

(0.7) $$e_G(\alpha x) = |\alpha| e_G(x), \quad \alpha \in K, \ x \in X;$$

(iv) *The following inequalities hold* :

(0.8) $$|e_G(x) - e_G(y)| \leq \|x - y\|, \quad x, y \in X;$$

(0.9) $$e_G(x) \leq \|x\|, \quad x \in X;$$

(v) *The mapping e_G is continuous on X endowed with strong topology.*

As a consequence of these properties we have the following corollary ([8], pp. 139):

0.2. COROLLARY. *Let $(X, \|\cdot\|)$ be a normed linear space, G a linear subspace in X and y, z two elements in X. Then the real function given by :*

$$\varphi(t) := e_G(y + tz), \quad t \in \mathbb{R} \tag{0.10}$$

is convex in $\mathbb{R}$. If $z \notin G$, then

$$\lim_{t\to\pm\infty} \varphi(t) = \infty. \tag{0.11}$$

Another property of the distance functional e_G referring to weak topology $\sigma(X, X^*)$ of X is embodied in the next theorem :

0.3. THEOREM. ([8], Theorem 6.6, pp. 139). *Let $(X, \|\cdot\|)$ be a normed linear space and G its linear subspace. Then e_G is lower semi-continuous in weak topology $\sigma(X, X^*)$ of X.*

For the proof of this result we send to [8] pp. 140.

Finally, we recall the following lemma which improve the definition of the distance functional.

0.4. LEMMA. *Let $(X, \|\cdot\|)$ be a normed linear space, G a linear subspace in X. Then :*

$$e_G(x) = \inf_{\substack{g\in G\\ \|g\|\leq 2\|x\|}} \|x - g\|, \quad x \in X. \tag{0.12}$$

2. Now, we shall present some concepts and results in best approximation theory which will be used in the sequel.

Let $(X, \|\cdot\|)$ be a normed linear space, G a proper linear subspace in X such that $\bar{G} \neq X$ and $x \in X \setminus \bar{G}$.

0.5. DEFINITION. The subspace G is called proximinal in X if for every $x \in X$ the set of all elements of best approximation in G to x :

$$\mathscr{P}_G(x) := \{g_0 \in G \mid \|x - g_0\| = \inf_{g\in G} \|x - g\|\} \subset G$$

is nonvoid.

If $\mathscr{P}_G(x)$ contains a unique element, then G will be called chebyshevian in X.

The following lemma of characterization [8] pp. 87 is important in what follows.

0.6. LEMMA. *Let $(X, \|\cdot\|)$, be a normed linear space and H a hyperplane in X containing the null element 0. Then H is proximinal iff there exists a vector $z \in X\setminus\{0\}$ such that $z \perp H$ in the sense of Birkhoff* (see Definition 0.8).

For other characterizations of proximinality we send to [8] pp. 86–95 and to the recent paper of the author [3].

3. Now, we shall give the notion of semi-inner product in the sense of Lumer and the connection of this concept with smooth normed linear spaces.

0.7. DEFINITION. ([5], [1] pp. 386). *Let X be a real or complex linear space. A mapping $(\,,\,)_L : X \times X \to K$ is called semi-inner product in the sense of Lumer or, for short, L-semi-inner product, if the following conditions are satisfied :*

(i) $(x + y, z)_L = (x, z)_L + (y, z)_L, \quad x, y, z \in X;$

(ii) $(\alpha x, y)_L = \alpha(x, y)_L, \quad \alpha \in K, \quad x, y \in X;$

(iii) $(x, x)_L > 0$ if $x \neq 0$

(iv) $|(x, y)_L|^2 \leqslant (x, x)_L\,(y, y)_L, \; x, y \in X;$

(v) $(x, \lambda y)_L = \bar{\lambda}(x, y)_L, \quad \lambda \in K, \quad x, y \in X.$

We note that the mapping $X \ni x \overset{\|\cdot\|}{\mapsto} (x, x_L)^{1/2} \in \mathbb{R}_+$ is a norm on X and the functional given by $X \ni x \overset{f_y}{\mapsto} (x, y)_L \in K$ is a continuous linear functional on the normed linear space $(X, \|\cdot\|)$ and $\|f_y\| = \|y\|$.

It is well known that a normed linear space is smooth iff there exists a unique L-semi-inner product which generates the norm or the L-semi-inner product which generates the norm is continuous i.e.,

$$\lim_{t\to 0} \mathrm{Re}\,(y, x + ty)_L = \mathrm{Re}(y, x)_L, \; x, y \in X \text{ ([1], pp. 387).} \tag{0.13}$$

0.8. DEFINITION. Let $(X, \|\cdot\|)$ be a normed linear space. The element $x \in X$ is called orthogonal in the sense of Birkhoff over $y \in X$ if

$$\|x + \lambda y\| \geqslant \|x\| \text{ for all } \lambda \in K. \tag{0.14}$$

We note that $x \perp y$.

In paper [7] (see also [1] pp. 401) Ioan Roşca introduced the concept of orthogonality in the sense of Lumer.

0.9. DEFINITION. Let $(\,,\,)_L$ be a L-semi-inner product on X and x, y two elements in X. The element $x \in X$ is said to be L-orthogonal over $y \in X$ if

$$(y, x)_L = 0. \tag{0.15}$$

We note that xLy (see [2], Definition 1.3).

The following lemma proved in [4] gives a characterization of L-orthogonality in terms of Birkhoff's orthogonality for smooth normed linear spaces.

0.10. LEMMA. ([4], Lemma 1.1). Let $(X, \|\cdot\|)$ be a smooth normed linear space and $(\,,\,)_L$ the L-semi-inner product which generates the norm $\|\cdot\|$. Then the following assertions are equivalent :

(i) $x \; L \; y;$

(ii) $x \perp y.$

Now, we shall give the concept of semi-inner product in the sense of Tapia associated to a real normed linear space.

0.11. DEFINITION. ([9], [1] pp. 389). Let $(X, \|\cdot\|)$ be a real normed linear space and let $f: X \to \mathbb{R}$, $f(x) := 1/2 \|x\|^2$, $x \in X$. Then the mapping

$$(x, y)_T := \lim_{t \downarrow 0} \frac{f(y + tx) - f(y)}{t}, \quad x, y \in X$$

is called semi-inner product in the sense of Tapia or, for short, T-semi-inner product.

In paper [2] we proved the following result.

0.12. LEMMA. ([2], Lemma 1.2). *Let $(X, \|\cdot\|)$ be a smooth normed linear space and let $(\ ,\ )_L$ be the L-semi-inner product which generates the norm $\|\cdot\|$. Then we have:*

$$(0.16) \qquad (y, x)_T = \operatorname{Re}(y, x)_L = \lim_{t \to 0} \frac{\operatorname{Re}(x, x + iy)_L - \|x\|^2}{t}$$

for every $x, y \in X$.

For others properties of L-semi-inner product and T-semi-inner product we send to [5], [7], [1] pp. 386–402, or to the recent papers of the author [2], [3] and [4].

1. Local representation of distance functional. Further on, we shall suppose that $(X, \|\cdot\|)$ is a smooth normed linear space over the real or complex number field and $(\ ,\ )_L$ will be the unique L-semi-inner product which generates the norm $\|\cdot\|$.

1.1. THEOREM. *Let H be a hyperplane containing the null element 0. Then the following sentences are equivalent:*

(i) *H is proximinal in X;*

(ii) *There exists an element $u \in X$, $\|u\| = 1$ such that*

$$(1.1) \qquad e_H(x) = |(x, u)_L|, \quad x \in X.$$

Proof. "(ii)⇒(i)". Firstly, we observe that $e_H(x) = 0$ for all $x \in H$ implies uLH. Since uLH is equivalent with $u \perp H$ (see Lemma 0.10) by Lemma 0.6 we deduce that H is proximinal in X.

"(i)⇒(ii)". If H is proximinal in X, then it is closed in X ([8], pp. 88) which implies that there exists a continuous linear functional $f \neq 0$ defined in X such that $H = \operatorname{Ker}(f)$.

On the other hand, by Lemma 0.6, there exists an element $w \in X \setminus \{0\}$ with the property: $w \perp H$ i.e., wLH.

It is easy to see that

$$f(x)w - f(w)x \in \operatorname{Ker}(f) = H, \quad x \in X,$$

and then

$$(f(x)w - f(w)x, w)_L = 0, \quad x \in H,$$

which gives by simple computation:

$$f(x) = \left(x, \frac{\overline{f(w)}}{\|w\|^2} w\right)_L, \quad x \in X.$$

Putting $v := \frac{\overline{f(w)}}{\|w\|^2} w$, we obtain:

$$f(x) = (x, v)_L, \quad x \in X \text{ and } \|f\| = \|v\|.$$

By Ascoli's theorem ([8] pp. 21), we have

$$d(x, H) = \frac{|f(x)|}{\|f\|}, \quad x \in X,$$

and then

$$e_H(x) = \frac{|(x, v)_L|}{\|v\|} = |(x, u)_L|, \quad x \in X$$

where $u := \frac{v}{\|v\|}$, and the theorem is proven.

1.2. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth normed linear space over the real number field and H a hyperplane in X containing the null element 0. Then the following conditions are equivalent:*

(i) *H is proximinal in X;*

(ii) *There exists an element $u \in X$, $\|u\| = 1$, such that*

$$(1.2) \qquad e_H(x) = |(x, u)_T|, \quad x \in X.$$

1.3. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth normed linear space over the complex number field and H a hyperplane in X containing the null element 0. Then the following assertions are equivalent:*

(i) *H is proximinal in X;*

(ii) *There exists an element $v \in X$, $\|v\| = 1$, such that*

$$(1.3) \qquad e_H(x) = [(x, v)_T^2 + (ix, v)_T^2]^{1/2}, \quad x \in X.$$

The proof follows by the previous theorem and by Lemma 0.12 of Introduction. We omit the details.

Now, we can give the theorem of local representation for the distance functional defined on an arbitrary linear subspace G in a smooth normed linear space $(X, \|\cdot\|)$.

1.4. THEOREM. *Let G be a proper linear subspace in X. Then the following sentences are equivalent:*

(i) *G is proximinal in X;*

(ii) *For every $x_0 \in X \setminus G$ there exists an element $u_{x_0} \in G \oplus [x_0]$, $\|u_{x_0}\| = 1$, such that*

$$(1.4.) \qquad e_G(x) = |(x, u_{x_0})_L| \quad x \in G \oplus [x_0].$$

Proof. Let $x_0 \in X \setminus G$ and $F_{x_0} := G \oplus [x_0]$. Then G is a hyperplane in F_{x_0} containing the null element 0.

"(i) $\Rightarrow$ (ii)". If G is proximinal in X, then G is proximinal in F_{x_0} (for every $x_0 \in X \setminus G$) and by implication "(i) $\Rightarrow$ (ii)" of the previous theorem, there exists an element $u_{x_0} \in F_x$, $\|u_{x_0}\| = 1$, such that relation (1.4) is valid.

"(ii) $\Rightarrow$ (i)". It is evident by same implication of Theorem 1.1 for the normed linear space F_{x_0} and by definition of proximinality.

The theorem is proven.

1.5. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth normed linear space over the real number field and G a proper linear subspace in X. Then the following sentences are equivalent:*

(i) *G is proximinal in X;*

(ii) *For every $x_0 \in X \setminus G$ there exists an element $u_{x_0} \in G \oplus [x_0]$, $\|u_{x_0}\| = 1$, such that:*

(1.5) $$e_G(x) = |(x, u_{x_0})_T|, \quad x \in G \oplus [x_0].$$

1.6. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth normed linear space over the complex number field and G a proper linear subspace in X. Then the following assertions are equivalent:*

(i) *G is proximinal in X;*

(ii) *For every $x_0 \in X \setminus G$ there exists an element $v_{x_0} \in G \oplus [x_0]$, $\|v_{x_0}\| = 1$, such that:*

(1.6) $$e_G(x) = [(x, v)_T^2 + (ix, v)_T^2]^{1/2}, \quad x \in G \oplus [x_0].$$

Further, we shall give some consequences which are important in applications.

1.7. CONSEQUENCES. 1. Let $(X, \|\cdot\|)$ be a smooth normed linear space and $(\ ,\ )_L$ be the L-semi-inner product which generates the norm $\|\cdot\|$. Then the following assertions are valid:

(i) If G is a linear subspace in X such that the unit sphere $S_G := \{g \in G \mid \|g\| \leq 1\}$ is weakly sequentially compact, then for every $x_0 \in X \setminus G$ there exists an element $u_{x_0} \in G \oplus [x_0]$, $\|u_{x_0}\| = 1$, such that (1.4) holds:

(ii) If G is a finite-dimensional subspace in X, then for every $x_0 \in X \setminus G$, there exists $u_{x_0} \in G \oplus [x_0]$, $\|u_{x_0}\| = 1$, such that (1.4) is valid;

(iii) If $(X, \|\cdot\|)$ is reflexive, then for every G a closed linear subspace in X and for any $x_0 \in X \setminus G$, there exists an element $u_{x_0} \in G \oplus [x_0]$ such that relation (1.4) is true.

The proofs follow by the above theorem and by Corollary 2.1, Corollary 2.2 and Corollary 2.4 from [8] pp. 91–92. We omit the details.

2. Let $(X, \|\cdot\|)$ be a normed linear space such that the normed dual $(X^*, \|\cdot\|)$ is smooth, $(\ ,\ )_L^*$ the L-semi-inner product which generates the norm and F a linear subspace in X^*. Then the following sentences are valid:

(i) If F is $\sigma(X^*, X)$ — closed in X^*, then for every $f_0 \in X^* \setminus F$ there exists a functional $g_{f_0} \in F \oplus [f_0]$, $\|g_{f_0}\| = 1$ such that

(1.7) $$e_F(f) = |(f, g)_L^*|, \quad f \in F \oplus [f_0].$$

(ii) If the unit sphere $S_F := F \cap S_{X^*}$ is $\sigma(X^*, X)$ — compact in X^*, then for every $f \in X^* \setminus F$: σ there exists a functional $g_{f_0} \in F \oplus [f_0]$, $\|g_{f_0}\| = 1$, such that (1.7) is valid,

(iii) If S_F is weak* sequentially compact set in X^*, then for every $f \in X^* \setminus F$ there exists $g_{f_0} \in F \oplus [f_0]$ such that (1.7) holds.

The proofs follow by Theorem 1.4 and by Corollary 2.5, 2.2 and Theorem 2.3 from [8] pp. 94–95. We omit the details.

Now, we shall point out a result that gives a characterization of chebyshevian subspaces in prehilbertian spaces.

1.8. THEOREM. *Let $(X; (\ ,\ ))$ be a prehilbertian space and G a proper linear subspace in X. Then the following conditions are equivalent:*

(i) *G is chebyshevian in X;*

(ii) *For every $x_0 \in X \setminus G$, there exists an element $u_{x_0} \in G \oplus [x_0]$, $\|u_{x_0}\| = 1$, such that*

(1.8) $$e_G(x) = |(x, u_{x_0})|, \quad x \in G \oplus [x_0].$$

Moreover, if $v_{x_0} \in G \oplus [x_0]$ is another element with the property (1.8), then there exists a scalar $\lambda \in K$ such that $v_{x_0} = \lambda u_{x_0}$ and $|\lambda| = 1$.

Proof. "(i) $\Rightarrow$ (ii)". The existence follows by Theorem 1.8.

Let us suppose that u_{x_0}, $v_{x_0} \in G^{\perp}$ (in $F_{x_0} := G \oplus [x_0]$) and since $\dim (G^{\perp}) = 1$ in F_{x_0}, there exists $\lambda \in K$ such that $v_{x_0} = \lambda u_{x_0}$.

Then by the previous relations we obtain:

$$e_G(x) = |(x, v_{x_0})| = |(x, \lambda u_x)| = |\lambda| \, |(x, u_{x_0})| = |(\lambda| e_G(x), \quad x \in G \oplus [x_0],$$

that implies $|\lambda| = 1$ and the first part of theorem is proven.

"(ii) $\Rightarrow$ (i)". It is evident by Theorem 1.4 and by the fact that prehilbertian spaces are strictly convex spaces (see [8], Corollary 3.3, pp. 102). We omit the details.

1.9. Corollary. *Let $(X; (\ ,\ ))$ be a prehilbertian space and H a hyperplane containing the null element 0. Then the following conditions are equivalent:*

(i) *H is chebyshevian in X;*

(ii) *There exists an element $u \in X$, $\|u\| = 1$ such that*

(1.9) $$e_H(x) = |(x, u)|, \quad x \in X.$$

In addition, if v is another element with the property (1.9), then there exists $\lambda \in K$ such that $v = \lambda u$, where $|\lambda| = 1$.

2. Characterizations of reflexivity. The distance functional e_G was utilized by V. N. Nikolski and I. S. Tiuremskih (see for example [8], Theorem 6.8, pp. 142 and Theorem 6.9, pp. 144) to characterize the reflexivity of general Banach spaces.

2.1. DEFINITION. The Banach space $(X, \|\cdot\|)$ has the (B_f)-property if for every increasing sequence of closed linear subspaces $(G_n)_{n\in N}$ in X such that:

(2.1)) $$\{0\} = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots, \quad n \in N;$$

and for any sequence of real numbers $(e_n)_{n\in N}$ satisfying the condition

(2.3) $$e_0 \geq e_1 \geq e_2 \geq \ldots \geq e_n \geq e_{n+1} = e_{n+2} = \ldots = 0, \quad n \in N$$

there exists an element $x \in X$ such that:

(2.3) $$e_{G_k}(x) = e_k, \quad k \in N.$$

Now, we can give Nikolski's theorem of reflexivity.

2.2. THEOREM. *A Banach space $(X, \|\cdot\|)$ has the (B_f)-property if and only if it is reflexive.*

For the decreasing sequences of linear subspaces we have the following theorem due to I. S. Tiuremskih.

2.3. THEOREM. *Let $(X; \|\cdot\|)$ be a Banach space. Then the following conditions are equivalent:*

(i) *For every decreasing sequence of closed linear subspaces $(G_n)_{n\in N}$ such that:*

(2.4) $$X = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \subset G_{n+1} \ldots, \quad n \in N;$$

and for any real sequence of $(e_n)_{n\in N}$ satisfying the assertion:

(2.5) $$0 = e_0 \leq e_1 \leq e_2 \leq \ldots \leq e_n \leq e_{n+1} \ldots \leq c < \infty$$

there exists an element $x \in X$ such that

(2.6) $$\|x\| = \lim_{n\to\infty} e_n \text{ and } e_{G_k}(x) = e_k, \quad k \in N;$$

(ii) *X is a reflexive Banach space.*

For the proof of these theorems we send to [8] pp. 142–144.

Now, by use of Theorem 1.1, we give a characterization of reflexivity for smooth Banach spaces in terms of the distance functional.

2.4. THEOREM. *Let $(X; \|\cdot\|)$ be a smooth Banach space and $(\ ,\ )_L$ of the L-semi-inner product which generates the norm $\|\cdot\|$. Then the following conditions are equivalent:*

(i) *X is a reflexive Banach space;*

(ii) *For any proper closed hyperplane H containing the null element 0, there exists a vector $u_H \in X$, $\|u_H\| = 1$, such that*

(2.7) $$e_H(x) = |(x, u_H)_L|, \quad x \in X.$$

Proof. "(i) ⇒ (ii)". It is evident by implication "(i) ⇒ (ii)" of Theorem 1.1, since every closed hyperplane in X is proximinal.

"(ii) ⇒ (i)". By the implication "(ii) ⇒ (i)" of Theorem 1.1 it follows that every closed hyperplane containing the null element is proximinal in X and by Corollary 2.4 of [8] pp. 92 it results that X is reflexive.

2.5. COROLLARY. *Let $(X, \|\cdot\|)$, be a smooth Banach space over the real number field. Then the following sentences are equivalent:*

(i) *X is a reflexive Banach space;*

(ii) *For any proper closed hyperplane H containing the null element 0, there exists a vector $u_H \in X$, $\|u_H\| = 1$, such that*

(2.8) $$e_H(x) = |(x, u_H)_T|, \quad x \in X.$$

2.6. COROLLARY. *Let $(X, \|\cdot\|)$ be a smooth Banach space over the complex number field. Then the following assertions are equivalent:*

(i) *X is a reflexive Banach space;*

(ii) *For any proper closed hyperplane H containing the null element 0, there exists a vector $v_H \in X$, $\|v_H\| = 1$, such that:*

(2.9) $$e_H(x) = [(x, v_H)_T^2 + (ix, v_H)_T^2]^{1/2}, \quad x \in X.$$

Finally, we give:

2.7. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth Banach space. Then the following conditions are equivalent:*

(i) *X is a reflexive Banach space;*

(ii) *For every G a proper closed linear subspace in X and for any $x_0 \in X \setminus G$ there exists an element $u_{G,x_0} \in G \oplus [x_0]$,*

$$\|u_{G,x_0}\| = 1, \text{ such that:}$$

(2.10) $$e_G(x) = |(x, u_{G,x_0})|_L, \quad x \in G \oplus [x_0],$$

where $(\ ,\ )_L$ denote the L-semi-inner product which generates the norm $\|\cdot\|$.

REFERENCES

1. Dincă G., *Metode variaţionale şi aplicaţii*, Ed. Tehnică, Bucureşti, 1980.
2. Dragomir S. S., *Representation of continuous linear functionals on smooth reflexive Banach spaces*, L'Analyse numérique et la théorie de l'approximation, **16**, 1(1987), pp. 19–28.
3. Dragomir S. S., *O caracterizare a elementului de cea mai bună aproximaţie în spaţii normate reale*, Stud. Cerc. Mat., **36** (1987), 497–508.
4. Dragomir S. S., *Representation of continuous linear functionals on smooth normed linear spaces*, L'Analyse numérique et la théorie de l'approximation (to appear).
5. Lumer G., *Semi-inner product spaces*, Trans. Amer. Math. Soc., **100** (1961), 29–43.
6. Nicolescu M., *Sur la meilleure approximation d'une fonction donnée par les fonctions d'une famille donnée*, Bul. Fac. Sti. Cernăuţi, **12** (1938), 120–128.
7. Roşca I., *Semi-produits scalaires et représentation du type Riesz pour les fonctionnélles linéaire et bornées sur les éspaces normés.*, C.R. Acad. Sci. Paris, **283** (**19**), 1976.
8. Singer I., *Cea mai bună aproximare în spaţii vectoriale normate prin elemente din subspaţii vectoriale*, Ed. Acad., Bucureşti, 1967.
9. Tapia R. A., *A characterization of inner product*, Proc. Amer. Math. Soc., **41** (1973), 569–574.

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