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REFINEMENTS OF SOME INEQUALITIES FOR ISOTONIC FUNCTIONALS

SEVER S. DRAGOMIR and JOSIP E. PEČARIĆ
(Timişoara) (Zagreb)

Abstract. The aim of this paper is to give some refinements to well-known inequalities of Čebyshev, Cauchy-Buniakovski-Schwartz, and Diaz-Metcalf for isotonic functionals defined on a subalgebra of real functions.

1. Introduction. Let us consider a nonempty set E and let $F(E)$ be the ordered algebra of all real functions defined on E. Further, we shall assume that L is a subalgebra of $F(E)$ so that :

(i) $f \in L$ implies $|f| \in L$;

(ii) $\mathbb{1} \in L$ where $\mathbb{1}(t) := 1$ for all $t \in E$.

We denote by $M_+(L)$ the set of all linear functional A defined on L ([7]). Recall that a functional A on L is isotonic, if $f \in L$, $f \geq 0$, implies $A(f) \geq 0$. For some inequalities involving linear isotonic functionals, see the papers [1], [2], [3], [4], [5] where further references are given.

For a given linear isotonic functional $A : L \to \mathbb{R}$, we denote by AC, $ACBS$, ADM the following mappings :

$$AC : L^3 \to \mathbb{R},\quad AC(f, g; h) := A(h)A(fgh) - A(fh)A(gh);$$

$$ACBS : L^2 \to \mathbb{R},\quad ACBS(f,g) := A(f^2)A(g^2) - [A(fg)]^2;$$

$$ADM : R^2 \times L^2 \to \mathbb{R};\quad ADM(m, M; f, g) := (M + m)A(fg) - - A(f^2) - mMA(g^2);$$

which will be called Čebyshev's, Cauchy-Buniakovski-Schwartz's respectively Diaz-Metcalf's functionals associated to A.

2. A Refinement of Cebyshev's Inequality. Before presenting the main results of this section, let us give a definition.

2.1. Definition. The real functions f, $g \in L$ are synchrone (asynchrone) on E if :

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq)\ 0 \tag{2.1}$$

for all $x, y \in E$.

The following theorem is an improvement of the well-known inequality of Čebyshev for the functionals set $M_+(L)$.

2.2. THEOREM. *Let $f, g, h \in L$, $h \geq 0$ and A a given isotonic functional defined on L.*

1. *If f, g are synchrone on E, then :*

$$(2.2)\qquad AC(f, g; h) \geq$$
$$\geq \max\{|AC(|f|, |g|; h)|, \ |AC(f, |g|; h)|, \ |AC(|f|, g; h)|\} \geq 0.$$

2. *If f, g are asynchrone on E, then :*

$$(2.3)\qquad AC(f, g; h) \geq$$
$$\geq \min\{-|AC(|f|, |g|; h)|, -|AC(f, |g|; h)|, -|AC(|f|, g; h)|\} \leq 0.$$

Proof. For every $x, y \in E$ we have :

$$|f(x) - f(y)| \geq ||f(x)| - |f(y)||, \quad |g(x) - g(y)| \geq ||g(x)| - |g(y)||$$

which implies :

$$|(f(x) - f(y))(g(x) - g(y))| \geq |(|f(x)| - |f(y)|)(|g(x)| - |g(y)|)|$$

respectively

$$|(f(x) - f(y))(g(x) - g(y))| \geq |(f(x) - f(y))(|g(x)| - |g(y)|)|$$

respectively

$$|(f(x) - f(y))(g(x) - g(y))| \geq |(|f(x)| - |f(y)|)(g(x) - g(y))|.$$

Let us now suppose that f, g are synchrone on E. Then

$$|(f(x) - f(y))(g(x) - g(y))| = f(x)g(x) + f(y)g(y) - f(x)g(y) - f(y)g(x)$$

is greater that :

$$||f(x)g(x)| + |f(y)g(y)| - |f(x)g(y)| - |f(y)g(x)||$$

respectively

$$||f(x)|(g(x)| + f(y)|g(y)| - f(x)|g(y)| - f(y)|g(x)||$$

respectively

$$||f(x)|g(x) + |f(y)|g(y) - |f(x)|g(y) - |f(y)|g(x)|.$$

By multiplying with $h(x)h(y) \geq 0$, $(x, y \in E)$ and applying the functional A with respect to x and then with respect to y and since $|A(k)| \leq \leq A(|k|)$ for $k \in L$, we obtain easy (2.2).

The proof of (2.3) is similar. We omit the details.

Putting $AC(f, g) := AC(f, g; \mathbb{1})$, we have the following corollary.

2.3. COROLLARY. *Let $f, g \in L$ and $A \in M_+(L)$.*

1. *If f, g are synchrone on E, then :*

$$(2.4)\qquad AC(f, g) \geq \max\{|AC(|f|, |g|)|, \ |AC(f, |g|)|, \ |AC(|f|, g)|\} \geq 0$$

2. *If f, g are asynchrone on E, then :*

$$(2.5)\qquad AC(f, g) \leq \min\{-|AC(|f|, |g|)|, -|AC(f, |g|)|, -$$
$$-|AC(|f|, g)|\} \leq 0.$$

2.4. REMARK. Putting $A((a_n)_{n\in N}) := \sum_{i=0}^{m} a_i$ or $A(f) := \int_a^b f(x)\,dx$ where $(a_n)_{n\in N}$ is a sequence of real numbers, $m \in \mathbb{N}$, and f is a Riemann integrable function on compact interval $[a, b]$, from the above results we can obtain some interesting inequalities for real numbers and for integrals, respectively.

3. A Refinement of Cauchy-Buniakovski-Schwartz's Inequality.

Further, we shall give an improvement of the well-known inequality of Cauchy-Buniakovski-Schwartz for isotonic functionals :

$$(3.1)\qquad A(f^2)A(g^2) \geq [A(fg)]^2$$

where $f, g \in L$.

3.1. THEOREM. *Let $f, g \in L$ and $A \in M_+(L)$. Then the following inequality*

$$(3.2)\qquad ACBS(f, g) \geq |A(f|f|)A(g|g|) - A(f|g|)A(|f|g)| \geq 0$$

is valid.

Proof. We start from the next elementary inequality :

$$|f(x)g(y) - f(y)g(x)| \geq ||f(x)||g(y)| - |f(y)||g(x)||, \quad x, y \in E.$$

By multiplying with $|f(x)g(y) - f(y)g(x)| \geq 0$, we obtain

$$(f(x)g(y) - f(y)g(x))^2 \geq$$
$$|(f(x)g(y) - f(y)g(x))(|f(x)||g(y)| - |f(y)||g(x)|)|; \quad x, y \in E.$$

This inequality gives :

$$f^2(x)g^2(y) - 2f(x)g(x)f(y)g(y) + f^2(y)g^2(x) \geq$$
$$|f(x)|f(x)|g(y)|g(y)| + f(y)|f(y)|g(x)|g(x)| -$$
$$|f(x)|g(x)f(y)|g(y)| - f(x)|g(x)||f(y)|g(y)|, \; x, \; y \in E.$$

Applying the functional A with respect to x and then with respect to y, and since $|A(k)| \leq A(|k|)$ for $k \in L$, we obtain easily (3.2).

3.2. COROLLARY. *Let $f \in L$ and $A \in M_+(E)$. Then the following inequality holds :*

$$(3.3)\qquad ACBS(f, 1) \geq |AC(f, |f|)| \geq 0.$$

4. A Refinement of Diaz-Metcalf's Inequality. J. B. Diaz and F. T. Metcalf proved (see [6] pp. 61) the following inequality for real numbers:

$$\sum_{i=1}^{n} a_i^2 + mM \sum_{i=1}^{n} b_i^2 \leqslant (M + m) \sum_{i=1}^{n} a_i b_i \tag{4.1}$$

where

$$m \leqslant \frac{a_i}{b_i} \leqslant M (i = \overline{1, n}). \tag{4.2}$$

Further, we shall improve this inequality for isotonic functionals defined on a subalgebra of $F(E)$ which satisfies the conditions (i)–(ii) of Introduction.

4.1. THEOREM. *Let $m, M \in \mathbb{R}$ and $f, g \in L$ such that:*

$$m \leqslant \frac{f(x)}{g(x)} \leqslant M, \ x \in E. \tag{4.3}$$

Then for all $A \in M_+(L)$, the following inequality holds:

$$ADM(m, M; f, g) \geqslant \max\{|ADM_1|, |ADM_2|, |ADM_3|\} \geqslant 0 \tag{4.4}$$

where

$$ADM_1 := ADM(|m|, |M|; |f|, |g|);$$

$$ADM_2 := (M + |m|)A(|f|g) - |m|MA(g|g|) - A(f|f|) \text{ and}$$

$$ADM_3 := (|M| + m)A(f|g|) - m|M|A(g|g|) - A(f|f|).$$

Proof. For every $x \in E$, we have:

$$H(x) := \left(M - \frac{f(x)}{g(x)}\right)\left(\frac{f(x)}{g(x)} - m\right) = \left|\left(M - \frac{f(x)}{g(x)}\right)\left(\frac{f(x)}{g(x)} - m\right)\right|.$$

Then we obtain:

$$H(x) \geqslant \left|\left(|M| - \frac{|f(x)|}{|g(x)|}\right)\left(\frac{|f(x)|}{|g(x)|} - |m|\right)\right|$$

respectively

$$H(x) \geqslant \left|\left(M - \frac{f(x)}{g(x)}\right)\left(\frac{|f(x)|}{|g(x)|} - |m|\right)\right|$$

respectively

$$H(x) \geqslant \left|\left(|M| - \frac{|f(x)|}{|g(x)|}\right)\left(\frac{f(x)}{g(x)} - m\right)\right|, \qquad x, y \in E.$$

By simple computation we obtain:

$$(M + m)f(x)\,g(x) - mMg^2(x) - f^2(x)$$

is greater than

$$|(|M| + |m|)\,|f(x)|\,|g(x)| - |m|\,|M|\,g^2(x) - f^2(x)|$$

respectively

$$|(M + |m|)\,|f(x)|g(x) - |m|M\,g(x)|g(x)| - f(x)|f(x)||$$

respectively

$$|(|M| + m)f(x)|g(x)| - m|M|g(x)|g(x)| - f(x)|f(x)||$$

for all $x \in X$.

Applying the functional A, we obtain easily (4.3).

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Department of Mathematics
Faculty of Technology
Ive Lole Ribara, 126/3
41000 Zagreb
Yugoslavia

Department of Mathematics
University of Timişoara
B-dul V. Pârvan, 4
1900 Timişoara
Romania