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APPROXIMATION OF BOUNDED NON-NEGATIVE FUNCTIONS BY POLYNOMIALS WITH POSITIVE COEFFICIENTS IN THE HAUSDORFF METRIC

V. AT. KOSTOVA
(Russe)

Abstract. An estimate in the Hausdorff metric for the approximation] of the non-negative bounded functions by polynomials with positive coefficients is obtained with participation of the modulus of continuity.

1. Introduction. We shall use the following notation :

R_Δ^M — the set of all real functions defined on the interval Δ, $\max\{|f(x)|,\ x\in\Delta\}\leqslant M,\ M>0$; $\omega(f;\delta)=\sup\{|f(x')-f(x'')|:|x'-x''|\leqslant\delta,\ x',x''\in\Delta\}$ — the modulus of continuity of $f\in R_\Delta^M$.

For every $f\in R_\Delta^M$, $\delta>0$ we define :

$$I(f;x)=\lim_{\delta\to 0} I(\delta,f;x);$$

$$S(f;x)=\lim_{\delta\to 0} S(\delta,f;x),$$

where

$$I(\delta,f;x)=\inf\{y: y\in f(t),\ t\in[x-\delta,x+\delta]\cap\Delta\};$$

$$S(\delta,f;x)=\sup\{y: y\in f(t),\ t\in[x-\delta,x+\delta]\cap\Delta\}.$$

The completed graph of $f\in R_\Delta^M$ is the segment function

$$\bar f(x)=[I(f;x),\ S(f;x)].$$

We define the Hausdorff distance between $f,\ g\in R_\Delta^M$ according to ([7] : pp. 34)

$$\tau(\Delta;f,g)=\max\{\sup_{A\in\bar f}\ \inf_{B\in\bar g}\ \rho(A,B),\ \sup_{A\in\bar g}\ \inf_{B\in\bar f}\ \rho(A,B)\},$$

where

$$\rho(A(x,y),\ B(\xi,\eta))=\max\{|x-\xi|,\ |y-\eta|\}.$$

We denote the Hausdorff distance between $I(\delta,f)$ and $S(\delta,f)$ as follows

$$\tau(\Delta,f;2\delta)=\tau(\Delta;\ I(\delta,f),\ S(\delta,f))$$

and call it a modulus of H-continuity of the function $f\in R_\Delta^M$.

Let P_n be the set of all algebraic polynomials of a degree at most n; P_n^* — the set of all $p_n \in P_n$ such that

$$p_n^*(x) = \sum_{i+j \leqslant n} a_{ij} x^i (1-x)^j, \ a_{ij} \geqslant 0.$$

The following estimates are established

In ([8]; [7], pp. 134) it is proved that there are absolute constants C_1^0 and C_2^0 such that for every $f \in R^M_{[a,b]}$

$$\inf\{\tau([a,b]; f, p_n), p_n \in P_n\} \leqslant (b-a)\left\{C_1^0 \frac{\ln n}{n} + C_2^0 \frac{M}{n}\right\}$$

is valid. Further in ([9]; [7], pp. 148) is shown that there are absolute constants C_3^0 and C_4^0 such that for every $f \in R^M_{[a,b]}$ the inequality

$$\inf\{\tau([a,b]; f, p_n), p_n \in P_n\} \leqslant (b-a)\left\{C_3^0 \frac{\ln(e + n\omega(f: n^{-1}))}{n} + C_4^0 \frac{1}{n}\right\}$$

holds.

For the approximation of the functions with polynomials $p_n \in P_n^*$ it is known that for every non-negative $f \in R^M_{[0,1]}$ there holds

$$\inf\{\tau([0,1]; f, p_n), p_n \in P_n^*\} = 0((n^{-1} \ln n)^{1/2}),$$

where the constant 0(1) depends only on M [3]. It is proved also that for every non-negative convex $f \in R^M_{[0,1]}$

$$\inf\{\tau([0,1]; f, p_n), p_n \in P_n^*\} = 0(n^{-1/2})$$

gets, where 0(1) depends only on M [4].

2. Main result

THEOREM. *Let us $f \in R^M_{[0,1]}$, $f \geqslant 0$ for $x \in [0,1]$. Then for sufficiently large n*

$$\inf\{\tau([0,1]; f, p_n), p_n \in P_n^*\} = 0\left(\left(\frac{\ln(e + n^{1/2}\,\omega(f; n^{-1/2}))}{n}\right)^{1/2}\right)$$

holds, where P_n^* is the set of all polynomials with positive coefficients of degree at most n.

Proof. This proposition shall be proved in two steps.

(1) Let l_0 be an arbitrary natural number; $x_k = k/8l_0$, ($k = 0,1,\ldots, 8l_0$) are points on $[0,1]$;

$$m_k = \min\{f(x) : |x_k - x| \leqslant 1/4\,l_0\};$$

$$M_k = \max\{f(x) : |x_k - x| \leqslant 1/4\,l_0\}.$$

We define a continuous function g such that:

$$g(x) = \begin{cases} m_{4k+2}, & x \in [x_{4k}, x_{4k+1}], \\ M_{4k+2}, & x \in [x_{4k+2}, x_{4k+3}], \\ \text{linear for } x \in [x_{4k+1}, x_{4k+2}], & x \in [x_{4k+3}, x_{4k+1}], \end{cases}$$

where $k = 0,1,\ldots, 2\,l_0 - 1$.

Obviously we have

$$\max\{g(x);\ x \in [0,1]\} = \max\{f(x);\ x \in [0,1]\}.$$

In view of the definition of Hausdorff distance and the modulus of H-continuity, function g has the properties:

(1) $$\tau([0,1]; f, g) \leqslant 1/2l_0;$$

(2) $$\tau([0,1]; g; \delta) \leqslant \delta, \text{ if } \delta \leqslant 1/8l_0.$$

The following inequality is also valid

(3) $$\omega(g; \delta) \leqslant 33\omega(f; \delta), \text{ if } 1/32l_0 \leqslant \delta \leqslant 1/4l_0.$$

Indeed, let us

$$d = \max\{\max[M_{4k+2} - m_{4k+2}, M_{4k+2} - m_{4k+6}] : k = 0,1,\ldots, 2l_0 - 1\}.$$

By the definition of the modulus of continuity there follows

$$\omega(f : 1/l_0) \geqslant d$$

On the other hand due to the definition of g

$$\omega(g; \delta) \leqslant d, \quad 0 < \delta \leqslant 1/4l_0$$

is true. Hence, if $1/32l_0 \leqslant \delta \leqslant 1/4l_0$ then

$$\omega(g; \delta) \leqslant \omega(f; 1/l_0) \leqslant (1 + 1/\delta l_0)\,\omega(f; \delta)$$
$$\leqslant 33\,\omega(f; \delta).$$

2) Let $B_n(g)$ be the Bernstein polynomials of function g:

$$B_n(g; x) = \sum_{\nu=0}^{n} g\left(\frac{\nu}{n}\right) p_{n,\nu}(x),$$

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

It is not difficult to see that $B_n(g) \in P_n^*$.

Now we shall prove that for sufficiently large n there holds

(4) $$\tau([0,1]; B_n(g), g) \leqslant c_0(n^{-1} \ln \gamma)^{1/2},$$

where

$$\gamma = e + n^{1/2}\,\omega(f; n^{-1/2}).$$

We shall use the following statement ([7], pp. 63).

Let $L(f)$ be linear, positive operator, defined on R^M_Δ. Then for every $\delta > 0$

$$\tau(\Delta; L(f), f) \leqslant \tau(\Delta, f; 2\delta) + \sup_{x\in\Delta} L(\omega(x, \delta, f); x) + M \sup_{x\in\Delta} |1 - L(1; x)|,$$

holds, where

$$\omega(x, \delta, f; t) = \begin{cases} 0 & t \in [x-\delta, x+\delta] \cap \Delta; \\ \omega(f; |x-t| - \delta), & t \in \Delta \setminus [x-\delta, x+\delta]. \end{cases}$$

Consequently for the Bernstein polynomial we have

$$\tau([0,1]; B_n(g), g) \leqslant \tau([0,1], g; 2\delta) + \delta^{-1}\omega(g; \delta) \sup_{0\leqslant x\leqslant 1} \sum_{\left|x-\frac{\nu}{n}\right|>\delta} \left|x - \frac{\nu}{n}\right| p_{n,\nu}(x). \tag{5}$$

First we shall prove

$$\delta^{-1} \sup_{0\leqslant x\leqslant 1} \sum_{\left|x-\frac{\nu}{n}\right|>\delta} \left|x - \frac{\nu}{n}\right| p_{n,\nu}(x) \leqslant C_1 \gamma^{-1}(\gamma^{-1}\ln\gamma)^{1/2}. \tag{6}$$

For this purpose we use the inequality

$$\delta^{-1} \sum_{\left|x-\frac{\nu}{n}\right|>\delta} \left|x - \frac{\nu}{n}\right| p_{n,\nu}(x) \leqslant \delta^{-m} \sum_{\left|x-\frac{\nu}{n}\right|>\delta} \left(x - \frac{\nu}{n}\right)^m p_{n,\nu}(x),$$

where m is an even number, $m \geqslant 2$.

It is well known that the function

$$\varphi_n(z; x) = e^{-zx}(1 - x + xe^{z/n})^n$$

can be presented as follows (see [1])

$$\varphi_n(z; x) = \sum_{m=0}^{\infty} W_{m,n}(x) z^m/m!,$$

where

$$W_{m,n}(x) = \sum_{\nu=0}^{n} \left(x - \frac{\nu}{n}\right)^m p_{n,\nu}(x).$$

But in accordance with the inequality of Cauchy

$$W_{m,n}(x) \leqslant m! \frac{M(R)}{R^m}, \tag{7}$$

where R is the radius of an arbitrary circle such that, the function $\varphi_n(z; x)$ is analytic in it, $M(R) = \max\{|\varphi_n(z; x)|;\ |z| = R,\ 0 \leqslant x \leqslant 1\}$.

Let us now set $R = (n \ln\gamma)^{1/2}$. We write $\varphi_n(z; x)$ as follows

$$\varphi_n(z; x) = \left\{1 + x(1-x) \sum_{k=2}^{\infty} \left(\frac{z}{n}\right)^k \cdot \frac{1}{k!} [(1-x)^{k-1} - (-x)^{k-1}]\right\}^n$$

and after some transformation we get

$$M(R) \leqslant \left\{1 + 1/4 \sum_{k=2}^{\infty} \frac{1}{k!} (n^{-1}\ln\gamma)^{k/2}\right\}^n \tag{8}$$

$$\leqslant \left\{1 + \frac{\ln\gamma}{4n} \sum_{k=2}^{\infty} \frac{1}{k!}\right\}^n$$

$$\leqslant \left\{1 + \frac{\ln\gamma}{5n}\right\}^n \leqslant \gamma^{1/5}.$$

By (7) and (8) we obtain

$$\max_{0\leqslant x\leqslant 1} |W_{m,n}(x)| \leqslant m!\, \gamma^{1/5}(n\ln\gamma)^{-m/2}$$

and taking into account the inequality of Stirling

$$m! \leqslant \sqrt{3m\pi}\, m^m e^{-m}$$

we have

$$\delta^{-1} \max_{0\leqslant x\leqslant 1} \sum_{\left|x-\frac{\nu}{n}\right|>\delta} \left|x - \frac{\nu}{n}\right| p_{n,\nu}(x) \tag{9}$$

$$\leqslant \sqrt{3m\pi} \left(\frac{m}{e\delta}\right)^m \gamma^{1/5}(n\ln\gamma)^{-m/2}.$$

Now we shall suppose that m is sufficiently large and set:

$$l_0 = [\sqrt{n}/8e^{24/10}(\ln\gamma)^{1/2}];$$

$$m = 2[\ln\gamma/2];$$

$$\delta = m\, e^{24/10}(n\ln\gamma)^{-1/2}.$$

($[a]$ — the most integer number $\leqslant a$).

We replace m and δ in (9) and after some calculation we have (6). But it is easy to verify that $1/32\, l_0 \leqslant \delta \leqslant 1/8 l_0$. Then (5) in view of (2), (3), (6) yield:

$$\tau([0,1]; B_n(g), g) \leqslant 2\delta + C_2\,\omega(f; \delta)\,\gamma^{-1}(\gamma^{-1}\ln\gamma)^{1/2}$$

$$\leqslant 2\delta + c_3\,\omega(f; \delta)\gamma^{-1}$$

$$\leqslant 2\delta + C_3(1 + \delta n^{1/2})\,\omega(f; n^{-1/2})\,\gamma^{-1}$$

$$\leqslant 2\delta + C_3[1 + e^{24/10}(\ln\gamma)^{1/2}] \cdot n^{-1/2}$$

$$\leqslant C_4(n^{-1}\ln\gamma)^{1/2}.$$

Further we use the properties of Hausdorff distance and obtain

$$\begin{aligned}\tau([0,1];\ B_n(g), f) &\leqslant \\ &\leqslant \tau([0,1]; f, g) + \tau([0,1];\ B_n(g),\ g) \\ &\leqslant 1/2l_0 + C_4(n^{-1}\ln\gamma)^{1/2} \leqslant c_5(n^{-1}\ln\gamma)^{1/2}.\end{aligned}$$

Finally, let us note that from [6] and the definition of Hausdorff distance follows that for the function

$$t(x) = |x - 1/2|,\ x \in [0,1]$$

the inequality

$$\inf\{\tau([0,1];\ t, p_n),\ p_n \in P_n^*\} \geqslant C_6 n^{-1/2}$$

is valid. But the last means that

$$\inf\{\tau([0,1];\ t, p_n),\ p_n \in P_n^*\} \geqslant C_7\left(\frac{\ln(e + n^{1/2}\omega(t;\ n^{-1/2}))}{n}\right)^{1/2}.$$

Hence (4) cannot be improved for $f \in R^M_{[0,1]}$.
The proof is completed.

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Violeta At. Kostova
TU "Angel Kanchev"
Center of Mathematics
7004 Russe, Bulgaria