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NOTE ON THE CUT VERTICES IN A REGULAR GRAPH

DĂNUȚ MARCU
(Bucharest)

Introduction. In this paper, we consider undirected graphs without loops or multiple edges. The maximum number of cut vertices in a connected graph on n vertices with m edges was determined in [2]. In [3], the maximum number $f(n, d)$ of cut vertices is determined in a connected graph on n vertices, which is regular of degree d, when $d \leqslant 4$. In [4], are obtained results which yield on upper bound for $f(n, d)$.

In this paper, we obtain a better bound for $f(n, d)$. We also determine the exact value of $f(n, d)$, when d is odd and the remainder obtained by dividing $n-2d-4$ by $d+1$ is less than $\frac{d+1}{2}$.

The Main results. Throughout this paper, we assume that d is an odd integer $\geqslant 3$, and n is an even integer $\geqslant d+1$.
Clearly, $f(d+1, d) = 0$. So, we assume that $n \geqslant d+3$, in what follows.

LEMMA. *If G is a connected graph on a vertices which is regular of degree d, then G has at most $b(n, d)$ blocks, where*

$$b(n, d) = \left[\frac{2n-d-5}{d+1}\right].$$

Proof. Since any pendant block of a graph which is regular of degree d has at least $d+2$ vertices, it follows that if $n \leqslant 2d+2$, then G is a block. So, let $n \geqslant 2d+4$.

Let k be the number of blocks in G and let $n_1, n_2, \ldots, n_k$ be the sizes of the blocks. Since G is connected, $n_i \geqslant 2$. Since G is regular of degree d, it follows that if $k \geqslant 2$, then the number of edges in the i-th block is at most $g(n_i)$, where

$$g(n_i) = \min\left\{\binom{n_i}{2}, \left[\frac{dn_i-1}{2}\right]\right\}.$$

We prove the lemma by showing that $k > b(n, d)$ leads to the contradiction $\frac{dn}{2} > \sum_{i=1}^{k} g(n_i)$.

Thus, let $k \geqslant b(n, d) + 1$. It is known (see [1]) that $\sum_{i=1}^{k} n_i = n + k - 1$. So, the average a of $n_1, n_2, \ldots, n_k$ is $\frac{n + k - 1}{k}$. Since $k \geqslant \frac{2n - 4 - (d+1)}{d+1} \geqslant \frac{n-1}{d-1}$ (the second inequality simplifies to $d^2 \leqslant n(d-3) + 4$), we have $a \leqslant d$. We now show that $n_1, n_2, \ldots, n_k$ can be replaced by $p_1, p_2, \ldots, p_k$, such that $\sum_{i=1}^{k} n_i = \sum_{i=1}^{p} p_i$, each p_i is either 2 or $\geqslant a$ and $\sum_{i=1}^{k} g(n_i) \leqslant \sum_{i=1}^{k} g(p_i)$. Let $3 \leqslant n_s < a$. Then, there is some $n_t \geqslant a$. If $d = 3$, then $a < 4$ and so $n_s = 3$. When $d = 3$ and n_t is odd, we replace n_s and n_t by $\frac{n_t + 3}{2}$ and $\frac{n_t + 3}{2}$. This does not increase $g(n_s) + g(n_t)$, since $3 + \frac{3n_t - 1}{2} \leqslant \left[\frac{3n_t + 7}{4}\right]$. Next, let $d \geqslant 5$ or $d = 3$, and n_t be even.

Then, we show that

$$\binom{n_s}{2} + g(n_t) \leqslant 1 + g(n_s + n_t - 2), \tag{1}$$

so that n_s and n_t may be replaced by 2 and $n_s + n_t - 2$. We consider several cases.

Case i. $n_t \leqslant d$ and $n_s + n_t - 2 \leqslant d$. Then, (1) reduces to $2(n_s + n_t) \leqslant n_s n_t + 4$, and this is evidently true.

Case ii. $n_t \leqslant d$ and $n_s + n_t - 2 \geqslant d + 1$. Then, we have

$$\binom{n_s}{2} + \binom{n_t}{2} \leqslant 1 + \frac{d(n_s + n_t - 2) - 2}{2},$$

so that (1) holds again.

Case iii. $n_t \geqslant d + 1$. Then, $n_s + n_t - 2 \geqslant d + 1$. If $d = 3$ and n_t is even, then it is easy to see that equality holds in (1). So, let $d \geqslant 5$. Replacing $\left[\frac{dn_t - 1}{2}\right]$ by $\frac{dn_t - 1}{2}$ and $\left[\frac{d(n_s + n_t - 2) - 1}{2}\right]$ by $\frac{d(n_s + n_t - 2) - 2}{2}$, inequality (1) becomes

$$(d - n_s)(n_s - 1) \geqslant d - 1. \tag{2}$$

If $n_s < d - 1$, then (2) is evidently true. If $n_s = d - 1$, then dn_t and $d(n_s + n_t - 2)$ are both even or both odd, and so (1) again holds. Now, let q be some $p_i \geqslant a$ and let

$$\alpha(q) = \frac{qk - (n + k - 1)}{n - k - 1} \tag{3}$$

Then, we show that

$$g(q) + \alpha(q) < \frac{dn}{2} \cdot \frac{\alpha(q) + 1}{k}. \tag{4}$$

We consider three cases, separately.

Case i. $q \leqslant d$. Then, (4) simplifies to

$$k(q - 1) - (n - 1)(q + 1) + dn > 0.$$

Since $q \geqslant a$, we have $k(q - 1) \geqslant n - 1$. Thus, to prove (4), we have to show that $(d - q)n + q > 0$, and this is evidently true.

Case ii. $q = d + 1$. Then, (4) becomes

$$qk(d - 2) + dq - 4 + n(4 - 2d) > 0.$$

Now, using $k \geqslant b(n, d) + 1$ and $4d - 6 > 0$, the above inequality is easily proved.

Case iii. $q \geqslant d + 2$. Then, $g(q) \leqslant \frac{dq - 1}{2}$. So, replacing $g(q)$ by $\frac{dq - 1}{2}$, (4) becomes

$$k(dq - 2q + 1) + 3n + dq - 2nd - 3 > 0.$$

Using $k \geqslant b(n, d) + 1$ and $q \geqslant d + 2$, it is enough to show that

$$n(2d^2 - 1) + d^3 - 3d^3 + 3 > 0,$$

and this is evidentently true.

Thus, whenever $q \geqslant a$, (4) is satisfied. Let $q_1, q_2, \ldots, q_\varepsilon$ be the numbers in $p_1, p_2, \ldots, p_k$ which are $\geqslant a$ and let β be the number of 2's in $p_1, p_2, \ldots, p_k$. Then,

$$\sum_{j=1}^{\varepsilon} \alpha(q_j) = \sum_{j=1}^{\varepsilon} \frac{q_j k - (n + k - 1)}{n - k - 1} = k - \varepsilon = \beta.$$

Hence, $\sum_{i=1}^{k} g(p_i) = \beta + \sum_{j=1}^{\varepsilon} g(q_j) = \sum_{j=1}^{\varepsilon} [\alpha(q_j) + g(q_j)] < \frac{dn}{2k} \sum_{j=1}^{\varepsilon} [\alpha(q_j) + 1] = \frac{dn}{2}$.

Since $\sum_{i=1}^{k} g(n_i) \leqslant \sum_{i=1}^{k} g(p_i)$, we arrive at a contradiction, which proves the lemma. □

THEOREM 1. *The maximum number of blocks in a connected graph on n vertices which is regular of degree d is $b(n, d)$ or $b(n, d) - 1$. If $n = 2(d + 2) + s(d + 1) + \delta$, where $0 \leqslant \delta < \frac{d+1}{2}$, then the maximum is $b(n, d)$.*

Proof. Let $n = 2(d + 2) + s(d + 1) + \delta$, where $0 \leqslant \delta < d + 1$. Define $G(n, d)$ to be the graph shown in figure 1, where $B_2, B_3, \ldots, B_{s+1}$

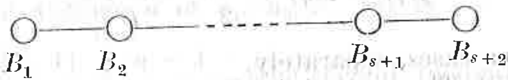

Figure 1

are complete graphs on $d + 1$ vertices with one edge removed, B_1 is a block on $d + 2$ vertices, B_{s+2} is a block on $d + 2 + \delta$ vertices and $B_1, B_2, \ldots, B_{s+2}$ are such that $G(n, d)$ is regular of degree d.

Clearly, $G(n, d)$ has $2s + 3$ blocks, and $b(n, d)$ is $2s + 3$ or $2s + 4$ according as $\delta < \dfrac{d+1}{2}$ or $\delta \geqslant \dfrac{d+1}{2}$. The theorem now follows from lemma. □

THEOREM 2. *The maximum number of cut vertices in a connected graph on n vertices which is regular of degree d is* $b(n, d)$-2 *or* $b(n, d)$-1. *If* $n = 2(d + 2) + s(d + 1) + \delta$, *where* $0 \leqslant \delta < \dfrac{d+1}{2}$, *then the maximum is* $b(n, d) - 1$.

Proof. The theorem is an immediate consequence of theorem 1, since in any connected graph the number of blocks is at least one more than the number of cut vertices. □

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Faculty of Mathematics,
University of Bucharest,
Academiei 14, 70109 – Bucharest,
Romania