MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 18, N° 2, 1989, pp. 101–103

ON AN INTEGRAL INEQUALITY

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In 1975 K.B. Stolarsky [5] introduced for positive real numbers a and b and for real parameter r the following mean value family:

$$L_r(a, b) = \left[\frac{a^r - b^r}{r(a - b)}\right]^{1/(r-1)}, \quad a \neq b, \ r \neq 0, 1.$$

Using limits $L_r(a, b)$ can also be defined for $r = 0, 1$ and $a = b$. We obtain

$$L_0(a, b) = \lim_{r\to 0} L_r(a, b) = \frac{b - a}{\log(b) - \log(a)}, \quad a \neq b,$$

$$L_1(a, b) = \lim_{r\to 1} L_r(a, b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}, \quad a \neq b,$$

and for all real r:

$$L_r(a, a) = \lim_{b\to a} L_r(a, b) = a.$$

A collection of remarkable properties of $L_r(a, b)$ were given in [1, Chapter VI].

In the last years, the identric mean

$$I(a, b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}$$

as well as the logarithmic mean

$$L(a, b) = \frac{b - a}{\log(b) - \log(a)}$$

have been investigated intensively by different authors [1, Chapter VI]. The logarithmic mean has noteworthy applications in some physical, chemical and economical problems [3]. This is certainly one reason for the great interest in this mean value.

Let $b > a > 0$. In a recently published note [4] a proof for the following proposition was given:

If the function $f \in C[a, b]$ is strictly increasing and if f has a logarithmically convex inverse function, then

$$(1) \qquad \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant f(I(a, b)).$$

If f is strictly decreasing, then the reversed inequality holds.

The aim of this paper is to present a lower bound for the integral mean $\frac{1}{b-a}\int_a^b f(x)\,dx$ under the conditions that $f \in C[a, b]$ is strictly increasing and that $1/f^{-1}$ is convex. (f^{-1} denotes the inverse function of f.) We will find an integral inequality in volving the logarithmic mean of a and b. Furthermore, we provide a new proof for (1).

We need the following

LEMMA. *Let $f \in C[a, b]$ and let F be continuous and convex such that $F \circ f$ is defined. Then*

$$(2) \qquad F\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leqslant \frac{1}{b-a}\int_a^b F(f(x))\,dx.$$

A proof for this proposition can be found in [2, Theorem 206].

A simple calculation reveals: If we set $F(x) = \log f^{-1}(x)$ in (2), then we get

$$f^{-1}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leqslant I(a, b)$$

which leads immediately to the theorem mentioned above. Now we prove a counterpart of (1).

THEOREM. *Let $f \in C[a, b]$ be a strictly increasing function. If $1/f^{-1}$ is convex, then*

$$f(L(a, b)) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx.$$

If f is strictly decreasing, then the reversed inequality holds.

Proof. Setting

$$F(x) = 1/f^{-1}(x)$$

in (2) we obtain

$$L(a, b) \leqslant f^{-1}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

which proves the Theorem.

Remark. The Theorem is in particular valid if we replace "$1/f^{-1}$ is convex" by "f^{-1} is logarithmically concave".

The following result was proved in [4]:

If $0 < p < 1$ and $0 \leqslant h \leqslant (1 - p)\min(a, b)$, then

$$(3) \qquad L_p(a + h, b + h) \leqslant I(a, b) + h.$$

An application of the Theorem leads to a counterpart of (3).

COROLLARY. *If $p \geqslant 0$ and $h \geqslant 0$, then*

$$(4) \qquad L(a, b) + h \leqslant L_p(a + h, b + h).$$

Proof. The function

$$f(x) = 1/(x + h),\ 0 < a \leqslant x \leqslant b,$$

is strictly decreasing. Further we have

$$f^{-1}(y) = -h + 1/y$$

and

$$\frac{d^2}{dy^2}\, 1/f^{-1}(y) = 2h\,[yf^{-1}(y)]^{-3} \geqslant 0.$$

From the Theorem we obtain inequality (4) with $p = 0$. Since $p \mapsto L_p(a + h, b + h)$ is increasing on $\mathbb{R}$ [5], we conclude that (4) is valid for all $p \geqslant 0$.

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Received 4. VII. 1989

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