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ON THE APPROXIMATE SOLUTION OF CERTAIN OPERATORIAL EQUATIONS

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0. In [2] we have presented a method for approximating the solution of the equation

$$P(x) = 0 \tag{0.1}$$

where $P : X \to Y$, X is a Banach space, Y is a linear normed space and 0 the zero of Y. This method is an improvement in a certain sense of the generalized method of chords and of the generalized Steffensen-type method. We assume that the operator P has the following property: there exists a real number a with $0 < a < 1$, such that for every x in a neighbourhood of a point $x_0 \in X$ we have the following inequality: $\| P(ax) \| \leqslant a \| P(x) \|$. In the same paper we have given examples with this property.

The present paper gives another variant of the method presented in [2]. We show that equation (0.1) has a unique solution in a neighbourhood of the initial approximant x_0 and that the order of convergence is 2.

In the first part of the present paper we assume that P is a continuous operator and $[u, v; P]$, $u \neq v$ [1] is a *symmetrical divided difference* of the operator P defined by $[u, v; P] : X^2 \to \mathscr{L}(X, Y)$, $[u, v; P](u - v) = P(u) - P(v)$. The symmetrical divided difference of second order of the operator $P : [u, v, w; P]$ defined successively has the following property:

$$[u, v, w; P](u - v) = [u, w; P] - [v, w; P]$$

For the approximate solving of equation (0,1) we define by reccurence the sequence (x_n), where

$$x_{n+1} = x_n - [x_n, ax_n; P]^{-1} P(x_n) = x_n - \Gamma_n P(x_n) \tag{0.2}$$

$n = 0, 1, 2, \ldots$, $x_0 \in X$ is the initial approximant and $P(x_0) \neq 0$.

1. THEOREM 1.1. *We suppose that there exists a point $x_0 \in X$, a real number a, with $0 < a < 1$ and the constants B, η_0, and M such that the following conditions hold*:

1° *for every* $u \in S[x_0, r] = \{x \in X : \|x - x_0\| \le r\}$ *there exists* $[u, au; P]^{-1}$ *and* $\|[u, au; P]^{-1}\| \le B$, *where*

$$r = max\left\{B\eta_0 \sum_{n=0}^{\infty} h^{2^n-1},\ (1-a)\|x_0\| + aB\eta_0 \sum_{n=0}^{\infty} h^{2^n-1}\right\};$$

2° $\|P(x_0)\| \le \eta_0$ *and* $\|P(ax)\| \le a\|P(x)\|$ *for every* $x \in S[x_0, r]$;

3° $\|[u, v, w; P]\| \le M$ *for every* $u, v, w \in S[x_0, r]$;

4° $h = aB^2 M\eta_0 < 1$.

Then: a) *the equation* (0,1) *has at least one solution* x^* *in the ball* $S[x_0, r]$;

b) *the equality* (0,2) *defines by recurrence the sequence* (x_n) *with* $x_n \in S[x_0, r]$ *and* $\lim_{n\to\infty} x_n = x^*$;

c) *for the error estimate we have the following inequality*:

$$\|x^* - x_n\| \le B\eta_0 h^{2^n-1} \sum_{k=0}^{\infty} h^{2^n(2^{k-1}-1)}$$

Proof. From formula (0.2) and the conditions 1° and 2° of the theorem it results

$$\|x_{n+1} - x_n\| \le B\|P(x_n)\|, \tag{1.1}$$
$$\|x_{n+1} - ax_n\| \le B\cdot a\|P(x_n)\|.$$

The first inequality is evident, while the second is obtained from the following equality:

$$x_{n+1} - ax_n = x_n - ax_n - [x_n, ax_n; P]^{-1} P(x_n) =$$
$$= [x_n, ax_n; P]^{-1}[x_n, ax_n; P](x_n - ax_n) - [x_n, ax_n; P]^{-1} P(x_n) =$$
$$= [x_n, ax_n; P]^{-1}[P(x_n) - P(ax_n) - P(x_n)] = -[x_n, ax_n; P]^{-1} P(ax_n)$$

because $x_n, ax_n \in S[x_0, r]$ for every $n \in N$, as we shall show.

From the definition of the divided differences, using equality (0.2) we get

$$[x_n, ax_n, x_{n+1}; P](x_{n+1} - ax_n)(x_{n+1} - x_n) = [x_n, x_{n+1}; P](x_{n+1} -$$
$$- x_n) - [x_n, ax_n; P](x_{n+1} - x_n) = P(x_{n+1}) - P(x_n) + P(x_n) = P(x_{n+1})$$

whence, using inequality (1.1) we obtain

$$\|P(x_{n+1})\| \le M\|x_{n+1} - ax_n\|\cdot\|x_{n+1} - x_n\| \le aB^2 M\|P(x_n)\|^2. \tag{1.2}$$

From inequality (1.1), using (1.2) we have

$$\|x_{n+1} - x_n\| \le B\eta_0 h^{2^n-1}$$

whence it results

$$\|x_{n+p} - x_n\| \le B\eta_0 h^{2^n-1}[1 + h^{2^n} + \dots + h^{2^n(2^p-1)}] < \tag{1.3}$$
$$< B\eta_0 h^{2^n-1} \sum_{k=0}^{\infty} h^{2^n}(2^k - 1)$$

Because the space X is a Banach space, there results the existence of the limit of the sequence (x_n). Let $x^* = \lim_{n\to\infty} x_n$.

From inequality (1.3) for $p \to \infty$ we obtain the error estimate formula of the theorem.

Using inequality (1.3) in the case $n = 0$ and $p = m$, we have

$$\|x_0 - x_m\| \le B\eta_0 [1 + h + h^3 + \dots + h^{2^m-1} + \dots] =$$
$$= B\eta_0 \sum_{m=0}^{\infty} h^{2^m-1} < \frac{5B\eta_0}{4(1-h^2)}$$

that is $x_m \in S[x_0, r]$ for every $m \in N$. Analogously there results that $ax_m \in S[x_0, r]$. Indeed, we have

$$\|x_0 - x_m\| \le \|x_0 - ax_0\| + \|ax_0 - ax_m\| \le (1-a)\|x_0\| + aB\eta_0 \sum_{m=0}^{\infty} h^{2^m-1}$$

From the relation $x_n - x_{n+1} = [x_n, ax_n; P]^{-1}P(x_n)$, using the continuity of P and the boundedness of the sequence $\|[x_n, ax_n; P]^{-1}\|$ we obtain that $x^* = \lim_{n\to\infty} x_n$ is the solution of equation (0.1).

THEOREM 1.2. *We suppose that there exists a point $x_0 \in X$, a real number a, with $0 < a < 1$, and the constants B, η_0 and M such that the following conditions hold*:

(i) *for every* $u, v \in S[x_0, r]$ *there exists* $[u, v; P]^{-1}$ *and* $\|[u, v; P]^{-1}\| = \|\Gamma\| \le B$;

(ii) $\|P(x_0)\| \le \eta_0$ *and* $\|P(ax)\| \le a\|P(x)\|$ *for every* $x \in S[x_0, r]$;

(iii) $\|[u, v, w; P]\| \le M$ *for every* $u, v, w \in S[x_0, r]$;

(iv) $h = aB^2M\eta_0 < 1$.

Then equation (0.1) *has a unique solution* x *in the ball* $S[x_0, r]$.

Proof. We observe that if the solutions x^* and ax^* are in the ball $S \in [x_0, r]$, then $x^* = ax^* = a^2x^* = \dots = 0$. Indeed, we have $\|x^* - ax^*\| = \|[x^*, ax^*; P]^{-1}[x^*, ax^*; P](x^* - ax^*)\| \le \|\Gamma\|\cdot\|P(x^*) - P(ax^*)\| = 0$.

Because the points x_0 and ax_0 are in the ball $S[x_0, r]$, the conditions of Theorem 1.1 are satisfied, hence equation (0.1) has at least one solution $x^* \in S[x_0, r]$ which is the limit of the sequence (x_n) defined by equation (0.2) and $x_n \in S[x_0, r]$ for every $n \in N$.

Let $\bar{x} \in S[x_0, r]$ be an arbitrary solution of equation (0,1) and $\bar{x} \neq ax^*$.

Let us define the auxiliary operator $F_n : X \to Y$ using the operator P by

$$F_n(x) = x - [x_n, ax_n;\ P]^{-1}\, P(x) = x - \Gamma_n\, P(x)$$

where x_n is the general term of the sequence (x_n).

The operator F_n has obviously the following properties:

$$F_n(\bar{x}) = \bar{x},\ F_n(x_n) = x_{n+1},\ [x_n, ax_n;\ F_n] = I - [x_n, ax_n;\ P]^{-1} \cdot [x_n, ax_n;\ P] = \tag{1.4}$$

$$= 0 \text{ and } [x_n, ax_n, \bar{x};\ F_n] = -\Gamma_n [x_n, ax_n;\ \bar{x};\ P]$$

for every $n \in N$.

Using the definition and the properties of the divided differences and relations (1.4) we have

$$[x_n, ax_n, \bar{x};\ F_n]\,(\bar{x} - ax_n)\,(\bar{x} - x_n) = [x_n, \bar{x};\ F_n]\,(\bar{x} - x_n) -$$

$$- [x_n, ax_n;\ F_n]\,(\bar{x} - x_n) = F_n(\bar{x}) - F_n(x_n) -$$

$$- [x_n, ax_n;\ F_n]\,(\bar{x} - x_n) = \bar{x} - x_{n+1} - 0 = \bar{x} - x_{n+1}$$

whence it results

$$\|\bar{x} - x_{n+1}\| \leqslant M \|\bar{x} - ax_n\| \cdot \|\bar{x} - x_n\| \tag{1.5}$$

From the evident equality

$$\bar{x} - ax_n = [\bar{x}, ax_n;\ P]^{-1}\, [\bar{x}, ax_n;\ P]\ [\bar{x} - ax_n] =$$

$$= [\bar{x}, ax_n;\ P]^{-1}\, [P(\bar{x}) - P(ax_n)] = -[\bar{x}, ax_n;\ P]^{-1}\, P(ax_n)$$

using the condition (ii) of Theorem 1.2 we obtain

$$\|\bar{x} - ax_n\| \leqslant B \|P(ax_n)\| \leqslant aB \|P(x_n)\|$$

Analogously it results that

$$\|\bar{x} - x_n\| \leqslant B\, \|P(x_n)\|$$

Therefore the inequality becomes

$$\|\bar{x} - x_{n+1}\| \leqslant aB^2\, M\, \|P(x_n)\|^2 \tag{1.6}$$

Using inequality (1.2), it results that

$$\|P(x_{n+1})\| \leqslant h^{2^n-1}\, \eta_0$$

and thus inequality (1.6) becomes

$$\|\bar{x} - x_{n+1}\| \leqslant h^{2^n}$$

From the above inequality it results that $\bar{x} = \lim\limits_{n\to\infty} x_n$. From the uniqueness of the limit in the space X we obtain $x^* = \bar{x}$. Therefore Theorem 1.2 was proved.

THEOREM 1.3. *If the conditions of Theorem 1.2 hold, then the iterative process* (0,2) *attached to equation* (0.1) *has the order of convergence* 2.

Proof. According to the definition of the order of convergence [3, pp. 175] we must prove that the following properties hold: there exists a constant $\rho > 0$, which is independent of n, such that

$$\rho\, \|P(x_0)\| < 1 \tag{1.7}$$

and the sequence (x_n) verifies the following inequality:

$$\|P(x_{n+1})\| \leqslant \rho\, \|P(x_n)\|^2$$

From condition (iv) of Theorem 1.2 it results that $aB^2M\,\|P(x_0)\| < 1$. Using the notation $\rho = aB^2M$, we obtain $\rho\,\|P(x_0)\| < 1$, which proves condition (1.7).

Above we proved that the following equality holds:

$$[x_n, ax_n, x_{n+1};\ P]\,(x_{n+1} - ax_n)\,(x_{n+1} - x_n) = P(x_{n+1}) \tag{1.9}$$

We prove that $x_{n+1} = \tilde{x}_{n+1}$ for every $n \geqslant 0$, where

$$\tilde{x}_{n+1} = ax_n - [x_n, ax_n;\ P]^{-1}\, P(ax_n)$$

Indeed, we have

$$\tilde{x}_{n+1} - x_{n+1} = ax_n - x_n + [x_n, ax_n;\ P]^{-1}\, P(x_n) - [x_n, ax_n;\ P]^{-1}\, P(ax_n) =$$

$$= [x_n, ax_n;\ P]^{-1}\, [x_n, ax_n;\ P]\,(ax_n - x_n) + [x_n, ax_n;\ P]^{-1}\, [P(x_n) -$$

$$- P(ax_n)] = [x_n, ax_n;\ P]^{-1} \{[x_n, ax_n;\ P]\,(ax_n - x_n) + P(x_n) -$$

$$- P(ax_n)\} = [x_n, ax_n;\ P]^{-1}\, [P(ax_n) - P(x_n) + P(x_n) - P(ax_n)] = 0$$

Therefore we have

$$x_{n+1} - x_n = -\Gamma_n\, P(x_n) \text{ and } x_{n+1} - ax_n = \Gamma_n\, P(ax_n)$$

Using the above equalities from relation (1.9) we obtain

$$P(x_{n+1}) = [x_n, ax_n, x_{n+1};\ P]\ \Gamma_n\, P(ax_n)\ \Gamma_n\, P(x_n)$$

whence it results that

$$\| P(x_{n+1}) \| \leqslant aB^2 M \| P(x_n) \|^2$$

or $\| P(x_{n+1}) \| \leqslant \rho \| P(x_n) \|^2$, thus condition (1.6) is verified.

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